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Some Results on Fixed Points Related to $\varphi - \psi$ Functions in JS – Generalized Metric Spaces

Eriola Sila[*,1], Sidita Duraj[2], Elida Hoxha[1]

[1]Department of Mathematics, Faculty of Natural Science, University of Tirana, Tirana, 1054, Albania

[2]Department of Mathematics, Faculty of Natural Science, University "Luigj Gurakuqi", Shkoder, 4000, Albania

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A B S T R A C T

In this paper are shown some new results on fixed point related to a $\varphi - \psi$ contractive map in *JS* – generalized metric spaces *X*. It proves that there exists a unique fixed point for a nonlinear map $f: X \to X$, using two altering distance functions. Furthermore, it gives some results which related to a couple of functions under some conditions in *JS* – generalized metric spaces. It provides a theorem where is shown that two maps $F, g: X \to X$ under a nonlinear contraction using ultra – altering distance functions ψ and φ, which are lower semi – continuous and continuous, respectively, have a coincidence point that is unique in *X*. In addition, there is proved if the maps F and g are weakly compatible then they have a fixed point which is unique in *JS* – generalized metric space. As applications, every theorem is illustrated by an example. The obtained theorems and corollaries extend some important results which are given in the references.

1. Introduction

The study of fixed point in metric space has an important role in Functional Analysis. It is developed in two directions during the years which are the improving contraction conditions and changing axioms of metric space with the intention to generalize fixed points results in this space. As a result, there are given many new spaces such as generalized metric space [1], cone metric space [2], rectangular metric space [3].

In [4], authors introduced the concept of *JS* – metric space. It generalizes well-known concepts of metric structures like metric spaces, *b* – metric spaces [5], metric - like spaces [6], etc. There are many articles which are worked on these spaces, like [7]-[10].

In [11], authors expanded the Banach contraction introducing the concept of weakly contractive function in Hilbert spaces.

Many authors have used these functions in many other spaces like [12]-[14].

In [15], authors provided some theorems related to fixed points of mappings in generalized metric spaces introducing the concept of altering distance functions.

In [16], authors generalized some theorems on fixed point for contractive functions of type Kannnan, Chatteria and Hardy – Rogers in generalized metric space.

This paper generalizes some important results on fixed point for weakly contractive functions and functions with altering distance between points, giving some new theorems on fixed point for $\varphi - \psi$ contractive functions in *JS* – generalized metric spaces.

Furthermore, it proves a theorem on common fixed points of two $\varphi - \psi$ contractive functions on *JS* – generalized metric spaces.

2. Preliminaries

Let X be a set which is not empty and D_{JS} be a non - negative function of cartesian product $X \times X$.

For each $\vartheta \in X$, let define the set

$$C(D_{JS}, X, \vartheta) = \{ \{x_n\} \subset X: \lim_{n\to\infty} D_{JS}(x_n, \vartheta) = 0\}$$

Definition 2.1 [4] The map D_{JS} that satisfies the conditions:

(D_{JS_1}) $D_{JS}(x, \tilde{x}) = 0$ yields $x = \tilde{x}$

*Corresponding Author: Eriola Sila, Roliaerjola.liftaj@fshn.edu.al

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(D_{JS_2}) $D_{JS}(x,\tilde{x}) = D_{JS}(\tilde{x},x)$

(D_{JS_3}) There exists a constant $\rho > 0$, such that for $x_n \in C(D_{JS}, X, \vartheta)$, it yields

$$D_{JS}(x,\tilde{x}) \le \rho \lim_{n\to+\infty} D_{JS}(x_n,\tilde{x})$$

for all $(x,\tilde{x}) \in X \times X$, is called JS – generalized metric.

The couple (X, D_{JS}) is called s JS – generalized metric pace.

In [4], authors showed that JS – generalized metric space are metric space, b – metric space, dislocated metric space, etc.

Remark 2.2 [4] As it can be seen in JS – generalized metric space, the map $D_{JS}(\tilde{x},\tilde{x})$ maybe not zero.

Remark 2.3 [4] When the set $C(D_{JS}, X, \vartheta)$ does not contain any element for all $\vartheta \in X$, the couple (X, D_{JS}) is JS – generalized metric space satisfying only $(D_{JS}1)$ and $(D_{JS}2)$.

Definition 2.4 [16] Let (X, D_{JS}) be a JS – generalized metric space and $\{x_n\}_{n\in N}$ is in X and $\vartheta \in X$. If $\{x_n\}_{n\in N}$ is in $C(D_{JS}, X, \vartheta)$, it is called D_{JS} – convergent to $\vartheta \in X$.

Remark 2.5 [16] If a constant sequence is D_{JS} – convergent to $c \in X$ then $D_{JS}(c,c) = 0$.

Proposition 2.6 [16] A necessary and sufficient condition that the set $C(D_{JS}, X, \vartheta)$ is not empty is that $D_{JS}(\vartheta,\vartheta) = 0$

Proposition 2.7 [4] Let the pair (X, D_{JS}) be a JS - metric generalized space and $\{x_n\}_{n\in N}$ in X and $\vartheta, \bar{\vartheta} \in X$. If $\{x_n\}_{n\in N}$ is D_{JS} – convergent to ϑ and $\bar{\vartheta}$ then $\vartheta = \bar{\vartheta}$.

Definition 2.8 [16] The sequence $\{x_n\}_{n\in N}$ in a JS – generalized metric space (X, D_{JS}) is called D_{JS} – Cauchy if $D_{JS}(x_n, x_m)$ converges to 0 when m, n tent to $+\infty$.

Definition 2.9 [16]A JS – generalized metric space (X, D_{JS}) is named D_{JS}– complete if each D_{JS} – Cauchy sequence in X is D_{JS} – convergent in X.

Example 2.10 Take $X = [0, c]$ where $c \in R$ and D_{JS} be a non - negative function of cartesian product $X \times X$, where

$$D_{JS}(x,\tilde{x}) = \begin{cases} \max\{x,\tilde{x}\}, & (x,\tilde{x}) \neq (0,0) \\ \dfrac{x}{2}, & (x,0) \\ \dfrac{\tilde{x}}{2}, & (0,\tilde{x}) \end{cases}$$

Since the map D_{JS} accomplishes (D_{JS_1}) and (D_{JS_2}) in trivially manner, it is a JS – generalized metric.

Consequently, it needs to verify that $\left(D_{JS_3}\right)$ is satisfied only for $x \in X$ that $D_{JS}(x,x) = 0$. But $D_{JS}(x,x) = 0$ implies $x = 0$.

If $(x_n)_{n\in N} \subset X$ converges to 0 then $\lim_{n\to+\infty} D_{JS}(x_n, 0) = 0$.

Taking $n \in N$ and $\tilde{x} \in X$,

$$D_{JS}(x_n,\tilde{x}) = \begin{cases} \max\{x_n,\tilde{x}\}, & for\ x_n \neq 0 \\ \frac{\tilde{x}}{2}, & for\ x_n = 0 \end{cases}.$$

As a result, the inequality $\frac{\tilde{x}}{2} < D_{JS}(x_n,\tilde{x})$ is true.

From this, it yields $D_{JS}(0,\tilde{x}) = \frac{\tilde{x}}{2} < \lim_{n\to+\infty} sup\, D_{JS}(x_n,\tilde{x})$.

So, (X, D_{JS}) is JS – generalized metric. However, it is not neither metric space nor dislocated metric space because it doesn't accomplish the third condition of their metric, for $x, \tilde{x} \in X - \{0\}$ and $x^* = 0$, because

$$D_{JS}(x,0) + D_{JS}(0,\tilde{x}) = \frac{x}{2} + \frac{\tilde{x}}{2} < \max\{x,\tilde{x}\} = D_{JS}(x,\tilde{x}).$$

Definition 2.11 [15] The function $\psi: R^+ \to R^+$ that satisfies the conditions:

1. ψ is non-decreasing and continuous
2. $\psi(s) = 0$ if and only if $s = 0$.
3. $\psi(s) \ge M s^{\mu}$, for every $s > 0$ where $M > 0, \mu > 0$ are constants

is called altering distance.

Definition 2.12 [15] A function $\psi: R^+ \to R^+$ that accomplishes the conditions:

1. ψ is non – decreasing
2. $\psi(s) > 0$ if $s > 0$ and $\psi(0) = 0$

is called ultra – altering distance.
The set of ultra-altering distances is denoted Ψ.

Let g, F two functions of X in itself.

Definition 2.13 [17] The point $\xi \in X$ which completes the equality $g(\xi) = F(\xi)$ is called coincidence point of g, F.

Definition 2.14 [18] The functions g, F that satisfy $gF\xi = Fg\xi$, for each ξ that is coincidence point of g and F, are called weakly compatible.

Definition 2.15 [18] The point $\xi \in X$, in which $\xi = F\,\xi = g\,\xi$, is called common fixed point of maps g, F.

Theorem 2.16 ([16], Corollary 5.5) Let f of set X in itself where$(X, D_{JS})$is D_{JS} – generalized metric space which is complete and

$$D_{JS}(fx, f\tilde{x}) \le \kappa max\{D_{JS}(x,\tilde{x}), D_{JS}(x,Tx), D_{JS}(\tilde{x},T\tilde{x}), \frac{D_{JS}(x,f\tilde{x}) + D_{JS}(\tilde{x},fx)}{2}\}$$

If there exists a $\varsigma \in X$ that $\delta(D_{JS}, f, \varsigma) < +\infty$, then the sequence $\{f^n\varsigma\}_{n\in N}$ D_{JS} – convergences to a point ϑ in X.

When $D_{JS}(\vartheta, f\vartheta) < +\infty$ then ϑ is fixed point of function f. Furthermore, for $\vartheta' \in X$ which is another fixed point of function f, such that $D_{JS}(\vartheta', f\vartheta') < +\infty$, it yields $\vartheta = \vartheta'$.

3. Main results

3.1. Fixed point results related to $\varphi - \psi$ contractions in JS – generalized metric spaces

Definitions 3.1.1 Define the set

$$\delta(D_{JS}, f, \varsigma) = \sup\{D_{JS}(f^l\varsigma, f^m\varsigma), l, m \geq 0\}$$

where $\varsigma \in X$.

Theorem 3.1.2 Let (X, D_{JS}) be JS – generalized metric space which is complete and $f: X \to X$ a function that satisfies the following condition:

$$\psi\left(D_{JS}(fx, f\tilde{x})\right) \leq \psi\left(M_{JS}(x, \tilde{x})\right) - \varphi(M_{JS}(x, \tilde{x})) \quad (3.1)$$

where

$$M_{JS}(x, \tilde{x}) = \text{maximum}\{D_{JS}(x, \tilde{x}), D_{JS}(x, fx), D_{JS}(\tilde{x}, f\tilde{x}), D_{JS}(x, f\tilde{x}), D_{JS}(\tilde{x}, fx)\},$$

for every x and $\tilde{x}$ from X and φ, ψ from Ψ, where ψ is continuous and φ is lower semi – continuous.

If there exists a $\varsigma \in X$ that $\delta(D_{JS}, f, \varsigma) < +\infty$, then the sequence $\{f^n\varsigma\}_{n\in N}$ D_{JS} – convergences to a point ϑ in X.

When $D_{JS}(\vartheta, f\vartheta) < +\infty$ then ϑ is fixed point of function f. Furthermore, for $\vartheta' \in X$ which is another fixed point of function f, such that $D_{JS}(\vartheta', f\vartheta') < +\infty$, it yields $\vartheta = \vartheta'$.

Proof. Let take $\varsigma \in X$ that $\delta(D_{JS}, f, \varsigma) < +\infty$. Applicating (3.1) for $i, j, n \in N$, it yields

$$\begin{aligned}
&\psi\left(D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma)\right) \\
&\leq \psi\left(M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma)\right) - \varphi\left(M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma)\right) \\
&\leq \psi\left(M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma)\right) \quad (3.2)
\end{aligned}$$

$$\begin{aligned}
&M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma) \\
&= \text{maximum}\left\{\begin{matrix} D_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma), D_{JS}(f^{n+i-1}\varsigma, f^{n+i}\varsigma), \\ D_{JS}(f^{n+j-1}\varsigma, f^{n+j}\varsigma), D_{JS}(f^{n+i-1}\varsigma, f^{n+j}\varsigma), \\ D_{JS}(f^{n+j-1}\varsigma, f^{n+i}\varsigma) \end{matrix}\right\} \\
&\leq \delta(D_{JS}, f, f^{n-1}\varsigma)
\end{aligned}$$

Consequently,

$$\psi\left(D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma)\right) \leq \psi\left(M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma)\right) \leq \psi\left(\delta(D_{JS}, f, f^{n-1}\varsigma)\right).$$

Using the non-decreasing monotony of ψ it results

$$D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma) \leq M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma) \leq \delta(D_{JS}, f, f^{n-1}\varsigma) \quad (3.3)$$

for every $i, j \in N, n \in N$.

$$\delta(D_{JS}, f, f^n\varsigma) = \sup\{D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma), i, j \in N\} \leq \delta(D_{JS}, f, f^{n-1}\varsigma).$$

As a result, the sequence $\{\delta(D_{JS}, f, f^n\varsigma)\}_{n\in N}$ is non – increasing and bounded.

Consequently, it D_{JS} – converges and $\lim_{n\to+\infty} \delta(D_{JS}, f, f^n\varsigma) = l \geq 0$.

In addition, from (3.3) the following inequality

$$\begin{aligned}
\delta(D_{JS}, f, f^n\varsigma) &\leq M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma) \\
&\leq \delta(D_{JS}, f, f^{n-1}\varsigma) \quad (3.4)
\end{aligned}$$

Taking the limit in (3.4) when $n \to +\infty$ and $i, j \in N$, it yields

$$\lim_{n\to+\infty} M_{JS}(f^{n+i-1}\varsigma, f^{n+j-1}\varsigma) = l \geq 0.$$

Furthermore, from

$$\delta(D_{JS}, f, f^n\varsigma) = \sup\{D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma), i, j \in N\},$$

it implies:

for each natural number k, there exist $i(k)$ and $j(k)$ that the inequality

$$l - \frac{1}{k} < D_{JS}(f^{n+i(k)}\varsigma, f^{n+j(k)}\varsigma) \leq l \quad (3.5)$$

holds.

Taking the limit in (3.5) when $k \to +\infty$, it yields

$$\lim_{n\to\infty} D_{JS}(f^{n+i(k)}\varsigma, f^{n+j(k)}\varsigma) = l.$$

Furthermore, since φ is lower semi – continuous and ψ is continuous, taking the limit in

$$\begin{aligned}
&\psi\left(D_{JS}(f^{n+i(k)}\varsigma, f^{n+j(k)}\varsigma)\right) \\
&\leq \psi\left(M_{JS}(f^{n+i(k)-1}\varsigma, f^{n+j(k)-1}\varsigma)\right) \\
&\quad - \varphi\left(M_{JS}(f^{n+i(k)-1}\varsigma, f^{n+j(k)-1}\varsigma)\right)
\end{aligned}$$

it implies $\psi(l) \leq \psi(l) - \varphi(l)$.

Consequently, $\varphi(l) = 0$. Since φ is in Ψ, it implies $l = 0$.

So

$$\lim_{n\to+\infty} \delta(D_{JS}, f, f^n\varsigma) = 0 \quad (3.6)$$

Since $\delta(D_{JS}, f, f^n\varsigma) = \sup\{D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma), i, j \in N\}$, it is true that

$$0 \le D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma) \le \delta(D_{JS}, f, f^n\varsigma),$$

for every $i, j \in N$.

Taking the limit when $i, j \to +\infty$, it implies

$$\lim_{i,j\to+\infty} D_{JS}(f^{n+i}\varsigma, f^{n+j}\varsigma) = 0 \tag{3.7}$$

Since the sequence $\{f^n\varsigma\}_{n\in N}$ is D_{JS} – Cauchy, there exists a point $\vartheta \in X$, that accomplishes:

$$\lim_{n\to+\infty} D_{JS}(f^n\varsigma, \vartheta) = 0 \tag{3.8}$$

Let prove now that $f(\vartheta) = \vartheta$.

$$D_{JS}(f\vartheta, \vartheta) \le \rho \lim_{n\to+\infty} D_{JS}(f\vartheta, f^{n+1}\varsigma)$$

Taking (3.1) for $x = \vartheta$ and $\tilde{x} = f^n\varsigma$, it implies

$$\psi(D_{JS}(f\vartheta, f^{n+1}\varsigma) \le \psi(M_{JS}(\vartheta, f^n\varsigma)) - \varphi\left(M_{JS}(\vartheta, f^n\varsigma)\right) \tag{3.9}$$

where

$$\begin{aligned} &M_{JS}(\vartheta, f^n\varsigma) \\ &= \text{maximum}\{D_{JS}(\vartheta, f^n\varsigma), D_{JS}(\vartheta, f\vartheta), D_{JS}(f^n\varsigma, f^{n+1}\varsigma), \\ &\qquad D_{JS}(\vartheta, f^{n+1}\varsigma), D_{JS}(f^n\varsigma, f\vartheta)\} \end{aligned}$$

Taking limit in $M_{JS}(\vartheta, f^n\varsigma)$ when $n \to +\infty$ and from (3.7) and (3.8), it yields

$$\begin{aligned} \lim_{n\to+\infty} M_{JS}(\vartheta, f^n\varsigma) &= \text{maximum}\{0, D_{JS}(\vartheta, f\vartheta), 0,0, D_{JS}(\vartheta, f\vartheta)\} \\ &= D_{JS}(\vartheta, f\vartheta) \end{aligned}$$

Taking limit in (3.9) it implies

$$\psi(D_{JS}(f\vartheta, \vartheta) \le \psi(D_{JS}(\vartheta, f\vartheta)) - \varphi\left(D_{JS}(\vartheta, f\vartheta)\right)$$

Consequently $\varphi\left(D_{JS}(\vartheta, f\vartheta)\right) = 0$ and

$$D_{JS}(\vartheta, f\vartheta) = 0 \tag{3.10}$$

From this, it yields $f\vartheta = \vartheta$.

If ϑ' is another point that $f\vartheta' = \vartheta'$ and $D_{JS}(\vartheta', \vartheta')$ is finite, then $D_{JS}(\vartheta', \vartheta') = 0$.

Indeed, applicating (3.1) for x and $\tilde{x}$ equal to ϑ', it yields $\psi(D_{JS}(\vartheta', \vartheta') = \psi(D_{JS}(f\vartheta', f\vartheta'))$

$$\le \psi(M_{JS}(\vartheta', \vartheta')) - \varphi\left(M_{JS}(\vartheta', \vartheta')\right)$$

where

$$\begin{aligned} &M_{JS}(\vartheta', \vartheta') \\ &= \text{maximum}\{D_{JS}(\vartheta', \vartheta'), D_{JS}(\vartheta', f\vartheta'), D_{JS}(\vartheta', f\vartheta'), \\ &\qquad D_{JS}(\vartheta', f\vartheta'), D_{JS}(\vartheta', f\vartheta')\} = D_{JS}(\vartheta', \vartheta') \end{aligned}$$

As a result $\psi(D_{JS}(\vartheta', \vartheta') \le \psi(D_{JS}(\vartheta', \vartheta')) - \varphi\left(D_{JS}(\vartheta', \vartheta')\right)$,

which implies $\varphi\left(D_{JS}(\vartheta', \vartheta')\right) = 0$ and $D_{JS}(\vartheta', \vartheta') = 0$.

Furthermore, $D_{JS}(\vartheta, \vartheta) = 0$, because $f(\vartheta) = \vartheta$.

Due to the fact $D_{JS}(\vartheta, \vartheta') = D_{JS}(f\vartheta, f\vartheta')$, it yields

$$\psi(D_{JS}(f\vartheta, f\vartheta') \le \psi(M_{JS}(\vartheta, \vartheta')) - \varphi\left(M_{JS}(\vartheta, \vartheta')\right) \tag{3.11}$$

where

$$\begin{aligned} M_{JS}(\vartheta, \vartheta') &= \text{maximum}\{D_{JS}(\vartheta, \vartheta'), D_{JS}(\vartheta, f\vartheta), D_{JS}(\vartheta', f\vartheta'), \\ &\qquad D_{JS}(\vartheta, f\vartheta'), D_{JS}(\vartheta', f\vartheta)\} \\ &= \text{maximum}\{D_{JS}(\vartheta, \vartheta'), 0,0, D_{JS}(\vartheta, \vartheta'), D_{JS}(\vartheta', \vartheta)\} \\ &= D_{JS}(\vartheta, \vartheta') \end{aligned}$$

From (3.11), it implies

$$\psi(D_{JS}(\vartheta, \vartheta') \le \psi(D_{JS}(\vartheta, \vartheta')) - \varphi\left(D_{JS}(\vartheta, \vartheta')\right).$$

As a result $\varphi\left(D_{JS}(\vartheta, \vartheta')\right) = 0$ and $D_{JS}(\vartheta, \vartheta') = 0$ and $\vartheta = \vartheta'$.

Remark 3.1.3 Theorem 3.1.2 generalizes Theorem 2.18 (Theorem 1.9) in [19].

Example 3.1.4 Let be $X = [0,1]$ and a non – negative D_{JS} of $X \times X$, where

$$D_{JS}(x, y) = \begin{cases} \text{maximum}\{x, \tilde{x}\}, & x \ne 0, \tilde{x} \ne 0 \\ \frac{x}{2}, & x \in X, \tilde{x} = 0 \\ \frac{\tilde{x}}{2}, & x = 0, \tilde{x} \in X \end{cases}.$$

(X, D_{JS}) is a JS – generalized metric space as it is shown in Example 2.10.

Let $f: X \to X$ be a map such that $f(x) = \frac{x^2}{2(1+x)}$ and the functions $\varphi, \psi \in \Psi$, $\varphi(s) = \frac{s}{2}$, $\psi(t) = \frac{3s}{2}$.

For $x \ne \tilde{x}$ and $x \ne 0, \tilde{x} \ne 0$

$$D_{JS}(f(x), f(\tilde{x})) = \text{maximum}\left\{\frac{x^2}{2(1+x)}, \frac{\tilde{x}^2}{2(1+\tilde{x})}\right\}.$$

Taking $x < \tilde{x}$, $D_{JS}(f(x), f(\tilde{x})) = \frac{\tilde{x}^2}{2(1+\tilde{x})}$

$$\begin{aligned} M_{JS}(x, \tilde{x}) = \text{maximum}\{&D_{JS}(x, \tilde{x}), D_{JS}(x, fx), D_{JS}(\tilde{x}, f\tilde{x}), \\ &D_{JS}(x, f\tilde{x}), D_{JS}(\tilde{x}, fx)\}, \end{aligned}$$

$$\begin{aligned} D_{JS}(x, \tilde{x}) = \text{maximum}\{x, \tilde{x}\} = \tilde{x}, D_{JS}(x, f(x)) \\ = \text{maximum}\left\{x, \frac{x^2}{2(1+x)}\right\} = x, \end{aligned}$$

$$D(\tilde{x}, f(\tilde{x}) = D\left(\tilde{x}, \frac{\tilde{x}^2}{2(1+\tilde{x})}\right) = \tilde{x},$$

$$D_{JS}(x, f(\tilde{x})) = \text{maximum}\left\{x, \frac{\tilde{x}^2}{2(1+\tilde{x})}\right\} = \left\{\begin{matrix} \frac{\tilde{x}^2}{2(1+\tilde{x})}, \\ x \end{matrix}\right.$$

$$D_{JS}(f(x), \tilde{x}) = \text{maximum}\left\{\tilde{x}, \frac{1}{2}\frac{x^2}{(1+x)}\right\} = \tilde{x}.$$

Case 1. If $D_{JS}(f(\tilde{x}), x) = \frac{1}{2}\frac{\tilde{x}^2}{(1+\tilde{x})}$, then

$$M_{JS}(x, \tilde{x}) = \text{maximum}\left\{\tilde{x}, x, \frac{1}{2}\frac{\tilde{x}^2}{(1+\tilde{x})}\right\}$$
$$= \text{maximum}\left\{\tilde{x}, \frac{1}{2}\frac{\tilde{x}^2}{(1+\tilde{x})}\right\} = \tilde{x}$$

$$\psi(D_{JS}(f(x), f(\tilde{x})) = \psi\left(\frac{\tilde{x}^2}{2(1+\tilde{x})}\right) = \frac{3}{2}\frac{\tilde{x}^2}{2(1+\tilde{x})} = \frac{3\tilde{x}^2}{4(1+\tilde{x})}.$$

$$\psi\left(M_{JS}(x, \tilde{x})\right) - \varphi\left(M_{JS}(x, \tilde{x})\right) = \psi(\tilde{x}) - \varphi(\tilde{x}) = \frac{3\tilde{x}}{2} - \frac{\tilde{x}}{2} = \tilde{x}$$

Since $\frac{3}{4}\frac{\tilde{x}^2}{(1+\tilde{x})} < \tilde{x}$, the condition of Theorem 3.1.2 is accomplished.

Case 2. If $D(x, f(\tilde{x}))$ is x, then

$$M_{JS}(x, \tilde{x}) = \text{maximum}\{\tilde{x}, x\} = \tilde{x}$$

$$\psi(D_{JS}(f(x), f(\tilde{x})) = \frac{3}{2}\frac{\tilde{x}^2}{2(1+\tilde{x})} = \frac{3\tilde{x}^2}{4(1+\tilde{x})}.$$

$$\psi(M(x, \tilde{x})) - \varphi(M(x, \tilde{x})) = \psi(\tilde{x}) - \varphi(\tilde{x}) = \frac{3\tilde{x}}{2} - \frac{\tilde{x}}{2} = \tilde{x}$$

Since $\frac{3}{4}\frac{\tilde{x}^2}{(1+\tilde{x})} < \tilde{x}$, the condition of Theorem 3.1.2 is completed.

Therefore, the function $f: X \to X$ has a fixed point which is 0.

Corollary 3.1.5. Let (X, D_{JS}) be a *JS* – generalized metric space which is complete and f a function of X in itself satisfying the condition

$$D_{JS}(fx, f\tilde{x}) \leq M_{JS}(x, \tilde{x}) - \varphi(M_{JS}(x, \tilde{x})),$$

where

$$M_{JS}(x, \tilde{x}) = \text{maximum}\left\{\begin{matrix} D_{JS}(x, \tilde{x}), D_{JS}(x, fx), D_{JS}(\tilde{x}, f\tilde{x}), \\ D_{JS}(x, f\tilde{x}), D_{JS}(\tilde{x}, fx) \end{matrix}\right\},$$

for every $x, \tilde{x} \in X$ and φ is from Ψ, and lower semi – continuous.

If there exists $\varsigma \in X$ that $\delta(D_{JS}, f, \varsigma) < +\infty$, then $\{f^n\varsigma\}_{n\in N}$ D_{JS} – convergences to a point ϑ in X.

When $D_{JS}(\vartheta, f\vartheta) < +\infty$ then ϑ is fixed point of function f. Furthermore, for $\vartheta' \in X$ which is another fixed point of function f, such that $D_{JS}(\vartheta', f\vartheta') < +\infty$, it yields $\vartheta = \vartheta'$.

Proof. Replacing $\psi(s) = s \in \Psi$ in above theorem, the proof is clear.

Corollary 3.1.6 Let (X, D_{JS}) be a *JS* – generalized metric space which is complete and f a function of X in itself satisfying the condition

$$D_{JS}(fx, f\tilde{x}) \leq \kappa M_{JS}(x, \tilde{x})$$

where

$$M_{JS}(x, \tilde{x}) = \text{maximum}\left\{\begin{matrix} D_{JS}(x, \tilde{x}), D_{JS}(x, fx), D_{JS}(\tilde{x}, f\tilde{x}), \\ D_{JS}(x, f\tilde{x}), D_{JS}(\tilde{x}, fx) \end{matrix}\right\},$$

for every $x, \tilde{x} \in X$ and $\kappa \in]0,1[$.

If there exists $\varsigma \in X$ that $\delta(D_{JS}, f, \varsigma) < +\infty$, then $\{f^n\varsigma\}_{n\in N}$ D_{JS} – convergences to a point ϑ in X.

When $D_{JS}(\vartheta, f\vartheta) < +\infty$ then ϑ is fixed point of function f. Furthermore, for $\vartheta' \in X$ which is another fixed point of function f, such that $D_{JS}(\vartheta', f\vartheta') < +\infty$, it yields $\vartheta = \vartheta'$.

Proof. Taking $\psi(s) = s$ and $\varphi(s) = (1-\kappa)s \in \Psi$ in inequality (3.1), the proof is clear.

Remark 3.1.7 Corollary 3.1.5 is a generalization of Theorem 2.16 which is the result of [16] as Corollary 5.5 and Corollary 3.8 in [20].

3.2. Common fixed point related to $\boldsymbol{\varphi - \psi}$ *contractive functions in JS – generalized metric space.*

Theorem 3.2.1 Let (X, D_{JS}) a *JS* – generalized metric space which is complete and $F, g: X \to X$ such that FX is a subset of gX and the set gX is closed in X and

$$\psi(D_{JS}(Fx, F\tilde{x}) \leq \psi\left(M_{JS}(gx, g\tilde{x})\right) - \varphi(M_{JS}(gx, g\tilde{x})) \quad (3.12)$$

where

$M_{JS}(gx, g\tilde{x}) =$
maximum $\{D_{JS}(gx, g\tilde{x}), D_{JS}(gx, Fx), D_{JS}(gy, F\tilde{x})\}$

for x and $\tilde{x}$ in X and φ and ψ in Ψ, ψ is continuous and φ is lower semi – continuous.

If there exist $\varsigma \in X$ such that $\delta(D_{JS}, F, \varsigma) < +\infty$ and $\delta(D_{JS}, g, \varsigma) < +\infty$ then the sequence $\{\gamma_n\}_{n\in N} = \{g\varsigma_{n+1}\}_{n\in N} = \{F\varsigma_n\}_{n\in N}$, where $\gamma_1 = F\varsigma = g\varsigma_1$, D_{JS} – converges to a point ϑ in X.

If $\delta(D_{JS}, F, \vartheta) < +\infty$ and $\delta(D_{JS}, g, \vartheta) < +\infty$ then the maps F, g have a unique coincidence point in X.

Furthermore, they have a unique common fixed point in X,

if they are weakly compatible.

Proof. Let take $\varsigma \in X$ such that $\delta(D_{JS}, F, \varsigma) < +\infty$ and $\delta(D_{JS}, g, \varsigma) < +\infty$. For $\varsigma \in X$, due to $FX \subset gX$ then $F\varsigma \in gX$. Consequently, there exists $\varsigma_1 \in X, g\varsigma_1 = F\varsigma$.

Reasoning in the same manner for ς_1 and so on, there can be defined the sequences $\{\varsigma_n\}_{n\in N}$ and $\{\gamma_n\}_{n\in N}$ such that

$$\gamma_n = g\varsigma_{n+1} = F\varsigma_n \tag{3.13}$$

for $n = 0, 1, 2, \ldots$

If there exists any $n \in N$, such that $\gamma_n = \gamma_{n+1}$ then $g\varsigma_{n+1} = F\varsigma_n = \gamma_n = \gamma_{n+1} = F\varsigma_{n+1}$.

So, ς_{n+1} is the required point for g and F.

Let suppose now that the terms of $\{\gamma_n\}$ are diferent from each other.

Consequently,

$$D_{JS}(\gamma_n, \gamma_{n+1}) > 0. \tag{3.14}$$

$$\psi\left(D_{JS}(\gamma_n, \gamma_{n+1})\right) = \psi\left(D_{JS}(F\varsigma_n, F\varsigma_{n+1})\right)$$

$$\leq \psi\left(M_{JS}(g\varsigma_n, g\varsigma_{n+1})\right) - \varphi\left(M_{JS}(g\varsigma_n, g\varsigma_{n+1})\right) \tag{3.15}$$

where

$$\begin{aligned} &M_{JS}(g\varsigma_n, g\varsigma_{n+1}) \\ &= \text{maximum}\{D_{JS}(g\varsigma_n, g\varsigma_{n+1}), D_{JS}(g\varsigma_n, F\varsigma_n), D_{JS}(g\varsigma_{n+1}, F\varsigma_{n+1})\} \\ &= \text{maximum}\{D_{JS}(g\varsigma_n, g\varsigma_{n+1}), D_{JS}(g\varsigma_n, g\varsigma_{n+1}), D_{JS}(g\varsigma_{n+1}, g\varsigma_{n+2})\} \\ &= \text{maximum}\{D_{JS}(g\varsigma_n, g\varsigma_{n+1}), D_{JS}(g\varsigma_{n+1}, g\varsigma_{n+2})\} \\ &= \text{maximum}\{D_{JS}(\gamma_{n-1}, \gamma_n), D_{JS}(\gamma_n, \gamma_{n+1})\} \end{aligned}$$

Case 1. $D_{JS}(g\varsigma_n, g_{\varsigma n+1}) = D_{JS}(\gamma_n, \gamma_{n+1})$, then replacing it in (3.15), it yields

$$\psi\left(D_{JS}(\gamma_n, \gamma_{n+1})\right) \leq \psi\left(D_{JS}(\gamma_n, \gamma_{n+1})\right) - \varphi\left(D_{JS}(\gamma_n, \gamma_{n+1})\right)$$

So $\varphi\left(D_{JS}(\gamma_n, \gamma_{n+1})\right) = 0$ and consequently $D_{JS}(\gamma_n, \gamma_{n+1}) = 0$, which is absurd due to (3.14).

Case 2. $M_{JS}(g\varsigma_n, g\varsigma_{n+1}) = D_{JS}(\gamma_{n-1}, \gamma_n)$ then replacing it in (3.15), it yields

$$\psi\left(D_{JS}(\gamma_n, \gamma_{n+1})\right) \leq \psi\left(D_{JS}(\gamma_n, \gamma_{n-1})\right) - \varphi\left(D_{JS}(\gamma_n, \gamma_{n-1})\right)$$

$$\leq \psi\left(D_{JS}(\gamma_n, \gamma_{n-1})\right) \tag{3.16}$$

Due to non-decreasing monotony of ψ, it implies

$$D_{JS}(\gamma_n, \gamma_{n+1}) \leq D_{JS}(\gamma_n, \gamma_{n-1}),$$

for all $n \in N$.

This shows that $\{D_{JS}(\gamma_n, \gamma_{n+1})\}_{n\in N}$ is non-increasing and lower bounded because $D_{JS}(\gamma_n, \gamma_{n+1}) \geq 0$.

Consequently, the sequence $\{D_{JS}(\gamma_n, \gamma_{n+1})\}_{n\in N}$ converges to $l \geq 0$, $\lim_{n\to+\infty} D_{JS}(\gamma_n, \gamma_{n+1}) = l$.

Taking the limit in (3.16) when $n \to +\infty$, it yields

$$\psi(l) \leq \psi(l) - \phi(l).$$

Consequently, $\varphi(l) = 0$ and $l = 0$.

So,

$$\lim_{n\to+\infty} D_{JS}(\gamma_n, \gamma_{n+1}) = 0 \tag{3.17}$$

Denote

$$c_k = \sup\{D_{JS}(\gamma_i, \gamma_j), i, j > k\} \tag{3.18}$$

c_k is finite for every $k \in N$ because $\delta(D_{JS}, F, \varsigma) < +\infty$ and $\delta(D_{JS}, g, \varsigma) < +\infty$.

Since the sequence $\{c_k\}_{k\in N}$ is non – increasing, lower bounded from zero, it is convergent to a point ζ,

$$\lim_{k\to+\infty} c_k = \zeta \geq 0 \tag{3.19}$$

From (3.17), it yields:

for each $p \in N$, there exist $i_p, j_p > p$, such that

$$c_p - \frac{1}{p} < D_{JS}\left(\gamma_{i_p}, \gamma_{j_p}\right) < c_p \tag{3.20}$$

Taking limit in (3.20) when $p \to +\infty$ and using (3.19), it yields

$$\lim_{p\to+\infty} D_{JS}\left(\gamma_{i_p}, \gamma_{j_p}\right) = \zeta \tag{3.21}$$

Knowing that $D_{JS}\left(\gamma_{i_p}, \gamma_{j_p}\right) = D_{JS}\left(F\varsigma_{i_p-1}, F\varsigma_{j_p-1}\right)$, it implies

$$\psi\left(D_{JS}\left(\varsigma_{i_p}, \varsigma_{j_p}\right)\right) = \psi\left(D_{JS}\left(F\varsigma_{i_p-1}, F\varsigma_{j_p-1}\right) \leq \right.$$

$$\psi\left(M_{JS}\left(g\varsigma_{i_p-1}, g\varsigma_{j_p-1}\right)\right) - \varphi\left(M_{JS}\left(g\varsigma_{i_p-1}, g\varsigma_{j_p-1}\right)\right) \tag{3.22}$$

where

$$\begin{aligned} &M_{JS}\left(g\varsigma_{i_p-1}, g\varsigma_{j_p-1}\right) \\ &= \text{maximum}\left\{\begin{matrix} D_{JS}\left(g\varsigma_{i_p-1}, g\varsigma_{j_p-1}\right), D_{JS}\left(g\varsigma_{i_p-1}, F\varsigma_{i-1}\right), \\ D_{JS}\left(g\varsigma_{j_p-1}, F\varsigma_{j_p-1}\right) \end{matrix}\right\} \\ &= \text{maximum}\left\{\begin{matrix} D_{JS}\left(\gamma_{i_p-2}, \gamma_{j_p-2}\right), D_{JS}\left(\gamma_{i_p-2}, g\varsigma_{i_p-1}\right), \\ D_{JS}\left(\gamma_{j_p-2}, \varsigma_{j_p-1}\right) \end{matrix}\right\} \end{aligned}$$

From (3.17) and (3.19), it yields

$$\lim_{p\to+\infty} M_{JS}\left(g\varsigma_{i_p-1}, g\varsigma_{j_p-1}\right) = \text{maximum}\{\zeta, 0,0\} = \zeta$$

Taking the limit in (3.22) when $p \to +\infty$ and from (3.21), it implies

$$\psi(\zeta) \leq \psi(\zeta) - \varphi(\zeta)$$

Consequently, $\varphi(\zeta) = 0$ and $\zeta = 0$.

So,

$$\lim_{k\to+\infty} c_k = 0 \qquad (3.23)$$

From (3.18) and (3.23), it yields $\lim_{i,j\to+\infty} D_{JS}(\gamma_i,\gamma_j) = 0$, which means that the sequence $\{\gamma_n\}_{n\in N}$ is $D_{JS}-$ Cauchy in X.

Since $\gamma_n = g\varsigma_{n+1} = F\varsigma_n$, it yields $\{\gamma_n\}_{n\in N} \subset g(X)$. Since the set $g(X)$ is closed, there is $\vartheta \in gX$ such that $\lim_{n\to+\infty} \gamma_n = \vartheta$. Consequently, $\lim_{n\to+\infty} g\varsigma_{n+1} = \lim_{n\to+\infty} F\varsigma_n = \vartheta$.

Since $\vartheta \in gX$ then there is $\mu \in X, g\mu = \vartheta$.

The other step is to show that $F\mu = \vartheta$.

Knowing $D_{JS}(\vartheta, F\mu) \leq \lim_{n\to+\infty} D_{JS}(F\varsigma_n, F\mu)$ and applying $x = \varsigma_n$ and $\tilde{x} = \mu$ at (3.12), it implies

$$\psi\left(D_{JS}(F\varsigma_n, F\mu)\right) \leq \psi\left(M_{JS}(g\varsigma_n, g\mu)\right) - \varphi\left(M_{JS}(g\varsigma_n, g\mu)\right) \qquad (3.24)$$

where

$$\begin{aligned} &M_{JS}(g\varsigma_n, g\mu) \\ &= \text{maximum}\{D_{JS}(g\varsigma_n, g\mu), D_{JS}(g\varsigma_n, F\varsigma_n), D_{JS}(g\mu, F\mu)\} \\ &= \text{maximum}\ \{D_{JS}(g\varsigma_n, \vartheta), D_{JS}(g\varsigma_n, F\varsigma_n), D_{JS}(\vartheta, F\mu)\} \end{aligned}$$

Since

$$\lim_{n\to+\infty} D_{JS}(g\varsigma_n, \vartheta) = 0$$

and

$$\lim_{n\to+\infty} D_{JS}(g\varsigma_n, F\varsigma_n) = \lim_{n\to+\infty} D_{JS}(\gamma_{n-1}, \gamma_n) = 0,$$

it implies $\lim_{n\to+\infty} M(g\varsigma_n, g\mu) = D_{JS}(\vartheta, F\mu)$.

Taking limit in (3.24) when $n \to +\infty$, it yields

$$\psi(D_{JS}(\vartheta, F\mu)) \leq \psi(\vartheta, F\mu) - \varphi(\vartheta, F\mu)$$

As a result $\varphi(\vartheta, F\mu) = 0$.

Consequently, $\vartheta = F\mu$.

The following step is to prove that ϑ is unique.

Suppose that there exists another point of coincidence $\vartheta' \neq \vartheta$ of g and F.

So, there exists $\mu_1 \in X$ which accomplishes $F\mu_1 = g\mu_1 = \vartheta'$

$$\psi\left(D_{JS}(\vartheta, \vartheta')\right) = \psi\left(D_{JS}(F\mu, F\mu_1)\right)$$

$$\leq \psi\left(M_{JS}(g\mu, g\mu_1)\right) - \varphi\left(M_{JS}(g\mu, g\mu_1)\right) \qquad (3.25)$$

For

$$\begin{aligned} &M_{JS}(g\mu, g\mu_1) \\ &= \text{maximum}\{D_{JS}(g\mu, g\mu_1), D_{JS}(g\mu, F\mu), D_{JS}(g\mu_1, F\mu_1)\} \\ &= \text{maximum}\{D_{JS}(\vartheta, \vartheta'), D_{JS}(\vartheta, \vartheta), D_{JS}(\vartheta', \vartheta')\} \qquad (3.26) \end{aligned}$$

$$D_{JS}(\vartheta, \vartheta) = D_{JS}(F\mu, F\mu)$$

$$\psi\left(D_{JS}(\vartheta, \vartheta)\right) = \psi\left(D_{JS}(F\mu, F\mu)\right)$$

$$\leq \psi\left(M_{JS}(g\mu, g\mu)\right) - \varphi\left(M_{JS}(g\mu, g\mu)\right) \qquad (3.27)$$

where

$M_{JS}(g\mu, g\mu) =$
$\text{maximum}\{D_{JS}(g\mu, g\mu), D_{JS}(g\mu, F\mu), D_{JS}(g\mu, F\mu)\} =$
$\max\{D_{JS}(\vartheta, \vartheta), D_{JS}(\vartheta, \vartheta), D_{JS}(\vartheta, \vartheta)\} = D_{JS}(\vartheta, \vartheta)$

Replacing $M_{JS}(g\mu, g\mu) = D_{JS}(\vartheta, \vartheta)$ in (3.27),

it implies

$$\psi\left(D_{JS}(\vartheta, \vartheta)\right) \leq \psi\left(D_{JS}(\vartheta, \vartheta)\right) - \varphi\left(D_{JS}(\vartheta, \vartheta)\right)$$

So, $\varphi\left(D_{JS}(\vartheta, \vartheta)\right) = 0$ and

$$D_{JS}(\vartheta, \vartheta) = 0 \qquad (3.28)$$

Using the same method, it can be proved that

$$D_{JS}(\vartheta', \vartheta') = 0 \qquad (3.29)$$

Replacing (3.28) and (3.29) in (3.26) and then in (3.25) it yields

$$M_{JS}(g\mu, g\mu_1) = D_{JS}(g\mu, g\mu_1) = D_{JS}(\vartheta, \vartheta')$$

$$\psi\left(D_{JS}(\vartheta, \vartheta')\right) \leq \psi\left(D_{JS}(\vartheta, \vartheta')\right) - \varphi\left(D_{JS}(\vartheta, \vartheta')\right)$$

From this, it implies $\varphi\left(D_{JS}(\vartheta, \vartheta')\right) = 0$ and $D_{JS}(\vartheta, \vartheta') = 0$. Consequently $\vartheta = \vartheta'$.

Furthermore, let prove that if g and F are weakly compatible then $F\vartheta = g\vartheta$.

From $D_{JS}(F\vartheta, \vartheta) = D_{JS}(F\vartheta, F\mu)$ and (3.12), it implies

$$\psi\left(D_{JS}(F\vartheta, \vartheta)\right) = \psi\left(D_{JS}(F\vartheta, F\mu)\right)$$

$$\leq \psi\left(M_{JS}(g\vartheta, g\mu)\right) - \varphi\left(M_{JS}(g\vartheta, g\mu)\right) \qquad (3.30)$$

$$\begin{aligned} &M_{JS}(g\vartheta, g\mu) \\ &= \text{maximum}\{D_{JS}(g\vartheta, g\mu), D_{JS}(g\vartheta, F\mu), D_{JS}(g\vartheta, F\mu)\} \\ &= \text{maximum}\{D_{JS}(F\vartheta, \vartheta), D_{JS}(F\vartheta, F\vartheta), D_{JS}(\vartheta, \vartheta)\} \end{aligned}$$

From (3.28), it is known that $D_{JS}(\vartheta, \vartheta) = 0$. Using the same method as in (3.28), it can be proved that $D_{JS}(F\vartheta, F\vartheta) = 0$.

Consequently, $M_{JS}(g\vartheta, g\mu) = D_{JS}(F\vartheta, \vartheta)$

Replacing this equality in (3.30), it implies

$$\psi\left(D_{JS}(F\vartheta, \vartheta)\right) \leq \psi\left(D_{JS}(F\vartheta, \vartheta)\right) - \varphi\left(D_{JS}(F\vartheta, \vartheta)\right)$$

From this $\varphi\left(D_{JS}(F\vartheta, \vartheta)\right) = 0$ and $D_{JS}(F\vartheta, \vartheta) = 0$.

So, $F\vartheta = \vartheta = g\vartheta$ and ϑ is unique.

Remark 3.2.2 Since *JS* – generalized metric spaces are metric spaces, *b* – metric spaces, dislocated metric spaces, partial metric spaces it implies that Theorem 3.2.1 is true in these spaces.

Example 3.2.3 Let be $X = [0, a]$, where $a \in R$ and D_{JS} the *JS* – generalized metric defined at Example 2.10 and the functions $\varphi, \psi \in \Psi$, $\varphi(s) = \frac{1}{5}s$, $\psi(s) = \frac{s}{2}$.

Let $g, F: X \to X$ two functions where $g(x) = \frac{x}{2}$ and $F(x) = \ln\left(1 + \frac{x}{4}\right)$.

The functions g, F complete the condition of Theorem 3.2.1.

Indeed,

$$D_{JS}(Fx, F\tilde{x}) = D_{JS}\left(\ln\left(1 + \frac{x}{4}\right), \ln\left(1 + \frac{\tilde{x}}{4}\right)\right)$$
$$= \text{maximum}\{\ln\left(1 + \frac{x}{4}\right), \ln\left(1 + \frac{\tilde{x}}{4}\right)\}$$

Since $x \neq \tilde{x}$, it is supposed that $x < \tilde{x}$ without restricting anything.

From monotony of logarithmic function

$$D_{JS}(Fx, F\tilde{x}) = D_{JS}\left(\ln\left(1 + \frac{x}{4}\right), \ln\left(1 + \frac{\tilde{x}}{4}\right)\right) = \ln\left(1 + \frac{\tilde{x}}{4}\right).$$

$$M_{JS}(g(x), g(\tilde{x})) = \text{maximum}\,\{D_{JS}(g(x), g(\tilde{x})), D_{JS}(g(x), F(x)), D_{JS}(g(\tilde{x}), F(\tilde{x}))\}$$

$$D_{JS}(g(\tilde{x}), g(x)) = \text{maximum}\{g(\tilde{x}), g(x)\} = \frac{\tilde{x}}{2}$$

$$D_{JS}(F(x), g(x)) = \max\{F(x), g(x)\} = \frac{1}{2}x.$$

$$D_{JS}(F(\tilde{x}), g(\tilde{x})) = \max\{F(\tilde{x}), g(\tilde{x})\} = \frac{\tilde{x}}{2}$$

$$M_{JS}(g(x), g(\tilde{x})) = \frac{1}{2}\tilde{x}$$

$$\psi(D_{JS}(Fx, F\tilde{x})) = \psi(\ln\left(1 + \tfrac{1}{4}\tilde{x}\right)) < \psi\left(\tfrac{1}{4}\tilde{x}\right) = \tfrac{1}{8}\tilde{x}.$$

$$\psi\left(M_{JS}(gx, g\tilde{x})\right) - \varphi\left(M_{JS}(gx, g\tilde{x})\right) = \psi\left(\frac{\tilde{x}}{2}\right) - \varphi\left(\frac{\tilde{x}}{2}\right) = \frac{3\tilde{x}}{20}$$

Since

$$\psi\left(M_{JS}(gx, g\tilde{x})\right) - \varphi\left(M_{JS}(gx, g\tilde{x})\right) - \psi\left(D_{JS}(Fx, F\tilde{x})\right) > \frac{3\tilde{x}}{20} - \frac{\tilde{x}}{8} = \frac{\tilde{x}}{40} > 0,$$

then g, F have a common fixed point 0.

Corollary 3.2.4 Let (X, D_{JS}) be a *JS* – generalized metric space which is complete and $g, F: X \to X$ such that gX is closed and FX is a subset of gX and

$$D_{JS}(Fx, F\tilde{x}) \leq M_{JS}(gx, g\tilde{x}) - \varphi(M_{JS}(gx, g\tilde{x})) \quad (3.31)$$

where

$M_{JS}(x, \tilde{x}) = \max\{D_{JS}(gx, g\tilde{x}), D_{JS}(gx, Fx), D_{JS}(g\tilde{x}, F\tilde{x})\}$, for $x, \tilde{x}$ in X and φ in Ψ is lower semi – continuous.

If there exist $\varsigma \in X$ such that $\delta(D_{JS}, F, \varsigma) < +\infty$ and $\delta(D_{JS}, g, \varsigma) < +\infty$ then the sequence $\{\gamma_n\}_{n\in N} = \{g\varsigma_{n+1}\}_{n\in N} = \{F\varsigma_n\}_{n\in N}$, where $\gamma_1 = F\varsigma = g\varsigma_1$, D_{JS} – converges to a point ϑ in X.

If $\delta(D_{JS}, F, \vartheta) < +\infty$ and $\delta(D_{JS}, g, \vartheta) < +\infty$ then the maps F, g have a unique coincidence point in X.

Furthermore, they have a unique common fixed point in X,

if they are weakly compatible.

Proof. Taking $\psi(s) = s$ in (3.12), the corollary is true.

Corollary 3.2.5 Let (X, D_{JS}) be a *JS* – generalized metric space which is complete and $g, F: X \to X$ such that gX is closed set and FX is a subset of gX in X and

$$D_{JS}(Fx, F\tilde{x}) \leq \kappa\max\{D_{JS}(gx, g\tilde{x}), D_{JS}(gx, F\tilde{x}), D_{JS}(g\tilde{x}, Fx)\}$$

where $\kappa \in]0,1[$ for x and $\tilde{x} \in X$.

If there exist $\varsigma \in X$ such that $\delta(D_{JS}, F, \varsigma) < +\infty$ and $\delta(D_{JS}, g, \varsigma) < +\infty$ then the sequence $\{\gamma_n\}_{n\in N} = \{g\varsigma_{n+1}\}_{n\in N} = \{F\varsigma_n\}_{n\in N}$, where $\gamma_1 = F\varsigma = g\varsigma_1$, D_{JS} – converges to a point ϑ in X.

If $\delta(D_{JS}, F, \vartheta) < +\infty$ and $\delta(D_{JS}, g, \vartheta) < +\infty$ then the maps F, g have a unique coincidence point in X.

Furthermore, they have a unique common fixed point in X,

if they are weakly compatible.

Proof. Taking $\psi(s) = s$, $\varphi(s) = (1 - \kappa)s$ in (3.12), the corollary is true.

Corollary 3.2.6 Let (X, D_{JS}) be a *JS* – generalized metric space which is complete and $F, g: X \to X$ such that $FX \subset gX$ and gX is closed set in X and

$$D_{JS}(Fx, F\tilde{x}) \leq \kappa_1 D_{JS}(gx, g\tilde{x}) + \kappa_2 D_{JS}(gx, Fx) + \kappa_3 D_{JS}(g\tilde{x}, F\tilde{x})$$

where $0 < \kappa_1 + \kappa_2 + \kappa_3 < 1$, for x and $\tilde{x}$ from X.

If there exist $\varsigma \in X$ such that $\delta(D_{JS}, F, \varsigma) < +\infty$ and $\delta(D_{JS}, g, \varsigma) < +\infty$ then the sequence $\{\gamma_n\}_{n\in N} = \{g\varsigma_{n+1}\}_{n\in N} = \{F\varsigma_n\}_{n\in N}$, where $\gamma_1 = F\varsigma = g\varsigma_1$, D_{JS} – converges to a point ϑ in X.

If $\delta(D_{JS}, F, \vartheta) < +\infty$ and $\delta(D_{JS}, g, \vartheta) < +\infty$ then the maps F, g have a unique coincidence point in X.

Furthermore, they have a unique common fixed point in X,

if they are weakly compatible.

Proof. Taking $\kappa = \kappa_1 + \kappa_2 + \kappa_3,\ \kappa \in \left]0, \frac{1}{3}\right[$, it implies

$$\kappa_1 D_{JS}(gx, g\tilde{x}) + \kappa_2 D_{JS}(gx, Fx) + \kappa_3 D_{JS}(g\tilde{x}, F\tilde{x}) \leq \kappa\left(D_{JS}(gx, g\tilde{x}) + D_{JS}(gx, Fx) + D_{JS}(g\tilde{x}, F\tilde{x})\right) \leq \kappa \cdot 3 \cdot M_{JS}(gx, g\tilde{x}).$$

Replacing $\varphi(s) = (1 - 3\kappa)s$ and $\psi(s) = s$ in (3.12), the corollary holds.

4. Conclusions

In this paper are given some theorems and corollaries on fixed points for weakly contractive functions and for contractive functions with altering distance between points in JS – generalized metric spaces. Furthermore, in it are proved some results related to common fixed points of two $\varphi - \psi$ contractive functions on *JS* – generalized metric spaces. Since *JS* – generalized metric spaces are metric spaces, *b* - metric spaces, dislocated metric spaces, partial metric spaces, it implies that all obtained results are true in above mentioned spaces. In additions, some important results given in this paper are generalizations of some known references. Concretely, Theorem 3.1.2 generalizes Theorem 1.9 in [19]. Corollary 3.1.5 is a generalization of Corollary 3.8 in [20] and Corollary 5.5 in [16].

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