

Multi-Step Iteration Algorithm of Total Asymptotically Quasi-Nonexpansive Maps

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ABSTRACT

In Banach spaces an iteration algorithm for two finite families of total asymptotically quasi-nonexpansive maps is introduced. Weak and strong convergence theorems of this algorithm to approximation common fixed points are proved by using suitable conditions. As well as, numerical example by using Mat-lab is given.

1. Introduction and Preliminaries

This paper was originally published in the Conference: 2018 International Conference on Advanced Science and Engineering (ICOASE), Iraq [1]. It is well known that the concept of asymptotically nonexpansive introduced by Goebel and Kirk [2]. Additionally, every asymptotically nonexpansive map of a Banach space has a fixed point is proved. In [3], Petryshyn and Williamson proved the weak and strong convergence for quasi-nonexpansive map by using a sufficient and necessary condition. Alber [4], a new class of asymptotically nonexpansive is introduced. As well as, approximating methods for finding their fixed points are studied. In 2014, G. S. Saluja [5] established the strong and weak convergence for approximating common fixed point for generalized asymptotically quasi-nonexpansive maps in a Banach space.

Very recently, In [6], the authors proposed an implicit iteration for two finite families of generalized asymptotically quasi-nonexpansive maps. As well as, some strong convergence theorems are established. It is useful to point out our findings in this area which appeared in [7].

Let B be a non-empty closed convex subset of a real Banach space M and T be a self-map of B . The set of all fixed points denoted by $F(T)$. A self-map T from B into M is called nonexpansive map [2] if

$$\|Ta - Tb\| \leq \|a - b\| \text{ for all } a, b \in B$$

and is called quasi- nonexpansive map [6] if $F(T) \neq \emptyset$ and

$$\|Ta - a^*\| \leq \|a - a^*\|$$

for all $a \in B$ and for all $a^* \in F(T)$.

A Banach space M is satisfying:

- "Opial's condition if for each sequence (a_n) in M , is weak convergence to a implies that $\lim_{n \rightarrow \infty} \inf \|a_n - a\| < \lim_{n \rightarrow \infty} \inf \|a_n - b\|$ for all $b \in M$ with $a \neq b$ ".
- "Kadec-Klee property if for each sequence (a_n) in M is weak convergence to (a) together with $\|a_n\|$ converges strongly to $\|a\|$ imply that (a_n) is strong convergence to a point $a \in M$ [7]".

The aim of this paper, an iterative scheme for two families of total asymptotically quasi-nonexpansive maps is established. The strong and weak convergence theorems of this scheme for approximation of common fixed points in Banach space by using suitable conditions are established. For this purpose, let us recall the following definitions and lemmas.

Definition (1.1): "A map T is named asymptotically nonexpansive [1] if there is a sequence (f_n) in $[0, +\infty)$ with $\lim_{n \rightarrow \infty} f_n = 0$ and $\|T^n a - T^n b\| \leq (1 + f_n)\|a - b\|$, for all $a, b \in B, n = 1, 2, \dots$

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If $F(T) \neq \emptyset$ there is a sequence (f_n) in $[0, +\infty)$ with $\lim_{n \rightarrow \infty} f_n = 0$ and $\|T^n a - a^*\| \leq (1 + f_n)\|a - a^*\|$, for all $a \in B, a^* \in F(T)$ and $n = 1, 2, \dots$. Therefore, T is named asymptotically quasi-nonexpansive map [10].

Definition (1.2): "A map T is named total asymptotically nonexpansive map [4] if there are null sequences of positive real numbers $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, n \geq 1$ and nondecreasing continuous function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$ such as $\forall a, b \in B$

$$\|T^n a - T^n b\| \leq \|a - b\| + f_n \psi \|a - b\| + g_n$$

T is named total asymptotically quasi-nonexpansive map if $F(T) \neq \emptyset$, there are null sequences of positive real numbers $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, n \geq 1, \sum_{n=1}^\infty f_n < \infty$ and $\sum_{n=1}^\infty g_n < \infty$, and nondecreasing continuous function $\psi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$ such as $\forall a \in B$ and $a^* \in F(T)$

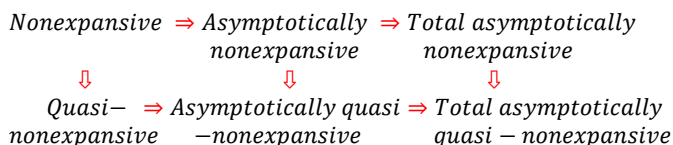
$$\|T^n a - a^*\| \leq \|a - a^*\| + f_n \psi \|a - a^*\| + g_n$$

If $g_n = 0, \forall n = 1, 2, \dots$ therefore T is asymptotically quasi-nonexpansive map.

Definition (1.3)[11]: "Let B be a nonempty closed convex subset of a Banach space M . A self-map T is named uniformly K -Lipschitzain if there exists a constant $K > 0$ such that $\|T^n a - T^n b\| \leq K \|a - b\|, \forall a, b \in B$.

Definition (1.4)[12]: "A map $T: B \rightarrow M$ is named demi-closed with respect to $b \in M$ if for each sequence (a_n) in $B, (a_n)$ is weak convergence to a and $T(a_n)$ is strong convergence to b . Hence $a \in B$ and $T(a) = b$. If $(I - T)$ is demiclosed which means if (a_n) is weak convergence to a in B and $(I - T)$ is strong convergence to 0. Therefore $(I - T)(a) = 0$."

Note: Now to explain the relation between the above definitions:



Lemma (1.5)[13]: "Let M be a uniformly convex Banach space and $0 < L \leq t_n \leq K < 1, \forall n \in N$. Presume that (a_n) and (b_n) are two sequences of M such as:

$$\lim_{n \rightarrow \infty} \|a_n\| \leq r, \lim_{n \rightarrow \infty} \|b_n\| \leq r \text{ and } \lim_{n \rightarrow \infty} \|t_n a_n + (1 - t_n) b_n\| = r$$

hold for some $r \geq 0$. Thus $\lim_{n \rightarrow \infty} \|a_n - b_n\| = 0$."

Lemma (1.6)[14]: "Let $(\mu_n)_{n=1}^\infty, (\sigma_n)_{n=1}^\infty$ and $(e_n)_{n=1}^\infty$ be sequences of positive numbers accomplishing the following inequality:

$$\mu_{n+1} \leq (1 + \sigma_n)\mu_n + e_n, \forall n \geq 1$$

if $\sum_{n=1}^\infty \sigma_n < \infty$ and $\sum_{n=1}^\infty e_n < \infty$, then (μ_n) is bounded and $\lim_{n \rightarrow \infty} \mu_n$ exists. In additional

if, $\liminf_{n \rightarrow \infty} \mu_n = 0$ then $\lim_{n \rightarrow \infty} \mu_n = 0$."

Lemma (1.7)[15]: "Let B be a nonempty convex subset of a uniformly convex Banach space. Therefore there exists a strictly nondecreasing continuous function $f: [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ such as for each Lipschitzain map $T: B \rightarrow B$ with Lipschitz constant K :

$$\begin{aligned} & \|tTa + (1 - t)Tb - T(ta + (1 - t)b)\| \\ & \leq Kf^{-1} \left(\|a - b\| - \frac{1}{K} \|Ta - Tb\| \right), \\ & \forall a, b \in B \text{ and } \forall t \in [0, 1]. \end{aligned}$$

Lemma (1.8)[12]: "Let M be a uniformly convex Banach space and its dual M^* accomplishing the Kadec-Klee property. Presume that (a_n) bounded sequence in M such as $\lim_{n \rightarrow \infty} \|ta_n + (1 - t)p_1 - p_2\|$ exists $\forall t \in [0, 1]$ and $p_1, p_2 \in W_w(a_n)$, thus $p_1 = p_2$."

2. Main Results

Let B be a nonempty closed convex subset of a Banach space M and $\{T_j, S_j, \forall j = 1, 2, \dots, k\}$ be two families of total asymptotically quasi-nonexpansive self-maps. We define the iteration algorithm (a_n) as follows:

$$\begin{aligned} & a_1 \in B \\ & a_{n+1} = (1 - \alpha_{jn})S_j^n a_n + \alpha_{jn}T_j^n b_{jn} \\ & b_{jn} = (1 - \alpha_{jn})S_j^n a_n + \alpha_{jn}T_j^n b_{(j-1)n} \\ & b_{(j-1)n} = (1 - \alpha_{(j-1)n})S_{j-1}^{n-1} a_n + \alpha_{(j-1)n}T_{j-1}^{n-1} b_{(j-2)n} \\ & \dots \\ & \dots \\ & b_{2n} = (1 - \alpha_{2n})S_2^n a_n + \alpha_{2n}T_2^n b_{1n} \\ & b_{1n} = (1 - \alpha_{1n})S_1^n a_n + \alpha_{1n}T_1^n b_{0n} \end{aligned} \tag{1}$$

Where $b_{0n} = a_n$ and $(\alpha_n)_{n=1}^\infty$ are sequences in $[0, 1]$.

Lemma (2.1): Let B be a nonempty closed convex subset of a normed space M and $T_j, S_j, j = 1, 2, \dots, k$ be two family of total asymptotically quasi-nonexpansive self-maps of B . Presume that $F(T_j, S_j) \neq \emptyset$ and the sequence (a_n) be as shown in step (1).

Then:

i- There are sequences (u_n) and (v_n) in $[0, \infty)$ such as $\sum_{n=1}^\infty u_n < \infty, \sum_{n=1}^\infty v_n < \infty$ and $\|a_{n+1} - a^*\| \leq (1 + u_n)^{j+1} \|a_n - a^*\| + v_n^{j+1}, \forall a^* \in F(T_j, S_j)$ and $\forall n$.

ii- There exist constants $J_1, J_2 > 0$ such as $\|a_{n+p} - a^*\| \leq J_1 \|a_n - a^*\| + J_2, \forall a^* \in F(T_j, S_j)$ and $n, p = 1, 2, \dots$

If there is $Z > 0$ such that $\psi(\lambda_j) \leq Z \lambda_j, j = 1, 2, \dots, k$.

Proof: i- Let $a^* \in F, u_n = \max_{1 \leq j \leq k} f_{jn}$ and $v_n = \max_{1 \leq j \leq k} g_{jn}$.

Now, we have

$$\begin{aligned} \|b_{1n} - a^*\| &= \|(1 - \alpha_{1n})S_1^n a_n + \alpha_{1n}T_1^n a_n - a^*\| \\ &\leq (1 - \alpha_{1n})\|S_1^n a_n - a^*\| + \alpha_{1n}\|T_1^n a_n - a^*\| \\ &\leq (1 - \alpha_{1n})\{\|a_n - a^*\| + f_{1n}\psi\|a_n - a^*\| + g_{1n}\} \\ &\quad + \alpha_{1n}\{\|a_n - a^*\| + f_{1n}\psi\|a_n - a^*\| + g_{1n}\} \\ &\leq (1 - \alpha_{1n})(1 + f_{1n}Z)\|a_n - a^*\| + (1 - \alpha_{1n})g_{1n} \\ &\quad + \alpha_{1n}(1 + f_{1n}Z)\|a_n - a^*\| + \alpha_{1n}g_{1n} \\ &\leq (1 + f_{1n}Z)\|a_n - a^*\| + g_{1n} \\ &\leq (1 + u_n)\|a_n - a^*\| + v_n \end{aligned}$$

Assume that $\|b_{jn} - a^*\| \leq (1 + u_n)^j \|a_n - a^*\| + v_n^j$

Therefore,

$$\|b_{(j+1)n} - a^*\| = \|(1 - \alpha_{(j+1)n})S_{j+1}^n a_n + \alpha_{(j+1)n}T_{j+1}^n b_j - a^*\|$$

$$\begin{aligned} &\leq (1 - \alpha_{(j+1)n}) \|S_{j+1}^n a_n - a^*\| \\ &\quad + \alpha_{(j+1)n} \|T_{j+1}^n b_j - a^*\| \\ &\leq (1 - \alpha_{(j+1)n})(1 + f_{(j+1)n} Z) \|a_n - a^*\| \\ &\quad + (1 - \alpha_{(j+1)n}) g_{(j+1)n} \\ &\quad + \alpha_{(j+1)n} (1 + f_{(j+1)n} Z) \|b_j - a^*\| \\ &\quad + \alpha_{(j+1)n} g_{(j+1)n} \\ &\leq (1 - \alpha_{(j+1)n})(1 + u_n) \|a_n - a^*\| + v_n \\ &\quad + \alpha_{(j+1)n} (1 + u_n)^j \|a_n - a^*\| \\ &\quad + \alpha_{(j+1)n} (1 + u_n) v_n^j \\ &\leq (1 + u_n)^{j+1} \|a_n - a^*\| + v_n^{j+1} \end{aligned}$$

Thus, by induction, we obtain

$$\|b_{jn} - a^*\| \leq (1 + u_n)^j \|a_n - a^*\| + v_n^j \tag{2}$$

for all $j = 1, 2, \dots, k$.

Now, by (2), we get

$$\begin{aligned} \|a_{n+1} - a^*\| &\leq (1 - \alpha_{jn}) \|S_j^n a_n - a^*\| + \alpha_{jn} \|T_j^n b_{jn} - a^*\| \\ &\leq (1 - \alpha_{jn})(1 + u_n) \|a_n - a^*\| + v_n \\ &\quad + \alpha_{jn} (1 + u_n)^j \|a_n - a^*\| \\ &\quad + \alpha_{jn} (1 + u_n) v_n^j \\ &\leq (1 + u_n)^{j+1} \|a_n - a^*\| + v_n^{j+1} \end{aligned}$$

ii- By using part (i), we get

$$\begin{aligned} \|a_{n+p} - a^*\| &\leq (1 + u_{n+p-1})^{j+1} \|a_{n+p-1} - a^*\| + v_{n+p-1}^{j+1} \\ &\leq e^{(1+u_{n+p-1})^{j+1} \|a_{n+p-1} - a^*\|} + e^{v_{n+p-1}^{j+1}} \\ &\leq e^{(j+1)u_{n+p-1} \|a_{n+p-1} - a^*\|} + e^{(j+1)v_{n+p-1}} \\ &\leq e^{(j+1)\sum_{k=1}^{n+p-1} u_k} \|a_n - a^*\| + e^{(j+1)\sum_{k=1}^{n+p-1} v_k} \\ &\leq J_1 \|a_n - a^*\| + J_2. \end{aligned}$$

Setting $J_1 = e^{(j+1)\sum_{k=1}^{n+p-1} u_k}$ and $J_2 = e^{(j+1)\sum_{k=1}^{n+p-1} v_k}$.

Lemma (2.2): Let B be a nonempty closed convex subset of a normed space M and $T_j, S_j, j = 1, 2, \dots, k$ be two families of total asymptotically quasi-nonexpansive self-maps of B . Presume $F(T_j, S_j) \neq \emptyset$ and (a_n) be as shown in step (1). Therefore, $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists for all $a^* \in F(T_i, S_i)$.

Proof: By Lemma (2.1.i)

$$\begin{aligned} \|a_{n+1} - a^*\| &\leq (1 + u_n)^{j+1} \|a_n - a^*\| + v_n^{j+1} \\ &\leq (1 + u_n) \|a_n - a^*\| + v_n \end{aligned}$$

and $\sum_{n=1}^{\infty} u_n < \infty, \sum_{n=1}^{\infty} v_n < \infty$. So by Lemma (1.6.i), we get $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists for all $a^* \in F(T_j, S_j)$.

Lemma (2.3): Let B be a nonempty closed convex subset of a Banach space M and $T_j, S_j, j = 1, 2, \dots, k$ be two families of Lipschitzain and total asymptotically quasi-nonexpansive self-maps of B . Let (a_n) be as shown in step (1). Therefore, for all $a_1^*, a_2^* \in F(T_j, S_j)$, the limit

$$\lim_{n \rightarrow \infty} \|ta_n + (1-t)a_1^* - a_2^*\| \text{ exists for all } t \in [0, 1].$$

If there is $Z > 0$ such that $\psi(\lambda_j) \leq Z\lambda_j, j = 1, 2, \dots, k$.

Proof: By using Lemma (1.6), we have $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists

$\forall a^* \in F(T_j, S_j)$ and (a_n) is bounded. Let

$$\gamma_n(t) = \|ta_n + (1-t)a_1^* - a_2^*\|, \forall t \in [0, 1].$$

Therefore, $\lim_{n \rightarrow \infty} a_n(0) = \|a_1^* - a_2^*\|$ and $\lim_{n \rightarrow \infty} a_n(1) =$

$\|a_n - a_2^*\|$ exist by Lemma (2.2).

Then, for $t \in [0, 1]$ and for all $a \in B$, we define the map

$R_n: B \rightarrow B$ by:

$$\begin{aligned} b_{1n} &= (1 - \alpha_{1n}) S_1^n a_n + \alpha_{1n} T_1^n b_{0n} \\ b_{2n} &= (1 - \alpha_{2n}) S_2^n a_n + \alpha_{2n} T_2^n b_{1n} \\ &\vdots \\ b_{jn} &= (1 - \alpha_{jn}) S_j^n a_n + \alpha_{jn} T_j^n b_{(j-1)n} \\ R_n a &= (1 - \alpha_{jn}) S_j^n a_n + \alpha_{jn} T_j^n b_{jn} \end{aligned}$$

Now,

$$\begin{aligned} \|R_n a - R_n c\| &\leq (1 - \alpha_{jn}) \|S_j^n a - S_j^n c\| + \alpha_{jn} \|T_j^n b_{jn} - T_j^n d_{jn}\| \\ &\leq (1 - \alpha_{jn})(1 + f_{jn} Z) \|a - c\| + (1 - \alpha_{jn}) g_{jn} \\ &\quad + \alpha_{jn} (1 + f_{jn} Z) \|b_{jn} - d_{jn}\| + \alpha_{jn} g_{jn} \\ &\leq (1 + u_n) \|a - c\| + v_n \\ &\leq (1 + f_n Z) \|a - b\| + g_n \end{aligned}$$

with $\sum_{n=1}^{\infty} u_n < \infty, \sum_{n=1}^{\infty} v_n < \infty$ and $s_n = 1 + u_n$, it follows that $s_n \rightarrow 1$ as $n \rightarrow \infty$.

Setting $W_{n,m} = R_{n+m-1} R_{n+m-2} \dots R_n$

and $b_{n,m} = \|W_{n,m}(ta_n + (1-t)a_1^* - (tW_{n,m}a_n + (1-t)a_1^*))\|$.

Thus,

$$\begin{aligned} \|W_{n,m} a - W_{n,m} b\| &= \left\| \begin{matrix} R_{n+m-1} R_{n+m-2} \dots R_n(a) \\ -R_{n+m-1} R_{n+m-2} \dots R_n(b) \end{matrix} \right\| \\ &\leq s_{n+m-1} \left\| \begin{matrix} R_{n+m-2} \dots R_n(a) \\ -R_{n+m-2} \dots R_n(b) \end{matrix} \right\| \\ &\quad + v_{n+m-1} \\ &\leq \prod_{j=n}^{n+m-1} s_j \|a - c\| + \sum_{j=n}^{n+m-1} v_j \\ &= A_n \|a - c\| + \sum_{j=n}^{n+m-1} v_j \end{aligned}$$

for all $a, c \in B$, where $A_n = \prod_{j=n}^{n+m-1} s_j, W_{n,m} a_n = a_{n+m}$ and $a^* = a^*$ for all $a^* \in F(T_i, S_i)$.

Hence,

$$\gamma_{n+m}(t) = \|ta_{n+m} + (1-t)a_1^* - a_2^*\|$$

$$\begin{aligned} &= \left\| \begin{matrix} tW_{n,m} a_n + ((1-t)a_1^* - a_2^* + W_{n,m}(ta_n + (1-t)a_1^*) - a_2^*) \\ + a^* - a^* \\ -W_{n,m}(ta_n + (1-t)a_1^*) - a_2^* \end{matrix} \right\| \\ &\leq b_{n,m} + \|W_{n,m}(ta_n + (1-t)a_1^*) - a_2^*\| \\ &\leq b_{n,m} + A_n \gamma_n(t) + \sum_{j=n}^{n+m-1} v_j \end{aligned}$$

By using Lemma (1.7), we have

$$\begin{aligned} b_{n,m} &\leq K f^{-1}(\|a_n - a^*\| - \frac{1}{K} \|W_{n,m} a_n - W_{n,m} a^*\|) \\ &\leq K f^{-1}(\|a_n - a^*\| - \frac{1}{K} (\|a_{n+m} - a^*\| \\ &\quad - \|W_{n,m} a_n - a^*\|)) \end{aligned}$$

and $(b_{n,m})$ converges uniformly to zero. Since $\lim_{n \rightarrow \infty} A_n =$

1 and $\lim_{n \rightarrow \infty} v_n = 0$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sup \gamma_{n+m} &\leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} b_{n,m} + \lim_{n \rightarrow \infty} \inf \gamma_n(t) \\ &= \lim_{n \rightarrow \infty} \inf \gamma_n(t) \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} \gamma_n(t)$ exists for all $t \in [0, 1]$.

Theorem (2.4): Let B be a nonempty closed convex subset of a uniformly convex Banach space M , $T_j, S_j, \forall j = 1, 2, \dots, k$ be two families of Lipschitz and total asymptotically quasi-nonexpansive self-maps of B and the sequence (a_n) be as shown in step (1). If there is $Z > 0$ such that $\psi((\lambda_j)) \leq Z\lambda_j, j = 1, 2, \dots, k$. Then, $\lim_{n \rightarrow \infty} \|T_j^n a_n - a_n\| = 0 = \lim_{n \rightarrow \infty} \|S_j^n a_n - a_n\|, \forall j$.

Proof: By Lemma (2.2), $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists. Assume that

$$\lim_{n \rightarrow \infty} \|a_n - a^*\| = e, \quad \forall e \geq 0.$$

If $e=0$, the proof is straight forward.

Now, suppose $e > 0$. We get

$$a_{n+1} = (1 - \alpha_{jn})S_j^n a_n + \alpha_{jn}T_j^n b_{jn}$$

and

$$e = \|a_{n+1} - a^*\| = \|(1 - \alpha_{jn})(S_j^n a_n - a^*) + \alpha_{jn}(T_j^n b_{jn} - a^*)\|.$$

Since $\|S_j^n a_n - a^*\| \leq K\|a_n - a^*\| \leq e$

and $\|T_j^n b_{jn} - a^*\| \leq K\|b_{jn} - a^*\| \leq K\{(1 + u_n)^j \|a_n - a^*\| + v_n^j\} \leq e$.

Therefore, by Lemma (1.5), we get

$$\lim_{n \rightarrow \infty} \|S_j^n a_n - T_j^n b_{jn}\| = 0$$

Next,

$$\begin{aligned} \|b_{jn} - a_n\| &\leq (1 - \alpha_{jn})\|S_j^n a_n - a_n\| + \alpha_{jn}\|T_j^n b_{(j-1)n} - a_n\| \\ &\leq (1 - \alpha_{jn})(1 + f_{jn}Z)\|a_n - a_n\|(1 - \alpha_{jn})g_{jn} \\ &\quad + \alpha_{jn}(1 + f_{jn}Z)\|b_{(j-1)n} - a_n\| + \alpha_{jn}g_{jn} \\ &\leq g_{jn} + \alpha_{jn}(1 - \alpha_{(j-1)n})(1 + f_{jn}Z) \\ &\quad \|S_{j-1}^n a_n - a_n\| + \alpha_{jn}\alpha_{(j-1)n}(1 + f_{jn}Z) \\ &\quad \|T_{j-1}^n b_{(j-2)n} - a_n\| \\ &\leq g_{jn} + \alpha_{jn}(1 - \alpha_{(j-1)n})(1 + f_{jn}Z)g_{jn} \\ &\quad + \alpha_{jn}\alpha_{(j-1)n}(1 + f_{jn}Z)(1 \\ &\quad + f_{(j-1)n}Z)\|b_{(j-2)n} - a_n\| \\ &\quad + \alpha_{jn}\alpha_{(j-1)n}(1 + f_{jn}Z)g_{(j-1)n} \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad < \infty \end{aligned}$$

Since $Z > 0, \sum_{n=1}^{\infty} f_{jn} < \infty$ and $\sum_{n=1}^{\infty} g_{jn} < \infty$, hence

$$\lim_{n \rightarrow \infty} \|b_{jn} - a_n\| = 0.$$

Then,

$$\begin{aligned} \|S_j^n a_n - a_n\| &\leq \|S_j^n a_n - T_j^n b_{jn}\| + \|T_j^n b_{jn} - a_n\| \\ &\leq \|S_j^n a_n - T_j^n b_{jn}\| + K\|b_{jn} - a_n\| \rightarrow 0 \\ &\quad \text{as } n \rightarrow \infty. \end{aligned}$$

And

$$\|T_j^n a_n - a_n\| \leq \|T_j^n a_n - S_j^n a_n\| + \|S_j^n a_n - a_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Theorem (2.5): Let B be a nonempty closed convex subset of a Banach space and $T_j, S_j, j = 1, 2, \dots, k$ be two families of total asymptotically quasi-nonexpansive self-maps of B . Presume that $F(T_j, S_j) \neq \emptyset$ and (a_n) be as shown in step (1) is strong convergence to a common fixed point of T_j and S_j iff

$$\lim_{n \rightarrow \infty} \inf d(a_n, F) = 0, \text{ where } d(a, F) = \inf_{a^* \in F} \|a - a^*\|.$$

Proof: To show $\lim_{n \rightarrow \infty} \inf d(a_n, F) = 0$ implies that (a_n) is strong convergence to a common fixed point of $T_j, S_j, j = 1, 2, \dots, k$, since by (2)

$$\begin{aligned} d(a_{n+1}, F) &\leq (1 + u_n)^{j+1}d(a_n, F) + v_n^{j+1} \\ &\leq (1 + u_n)d(a_n, F) + v_n \end{aligned}$$

By Lemma (1.6), we get $\lim_{n \rightarrow \infty} a_n$ exists and $\lim_{n \rightarrow \infty} \inf a_n = 0$.

Hence, $\lim_{n \rightarrow \infty} a_n = 0$.

Next, to prove the sufficiency, firstly show that (a_n) Cauchy sequence. By using Lemma (2.1.ii), we get

$$\|a_{n+p} - a^*\| \leq J_1 \|a_n - a^*\| + J_2 \tag{3}$$

$\forall a^* \in F(T_j, S_j), n = p = 1, 2, \dots$

Since $\lim_{n \rightarrow \infty} a_n = 0, \forall \epsilon > 0, \exists N$ such that

$$d(a_n, F) \leq \frac{\epsilon}{3J_1} - \frac{J_2}{J_1}, \quad \forall n \geq N$$

therefore, there is $l \in F(T_i, S_i)$ such that

$$\|a_N - l\| \leq \frac{\epsilon}{2J_1} - \frac{J_2}{J_1} \tag{4}$$

From (3) and (4), $\forall n \geq N$, we get

$$\begin{aligned} \|a_{n+p} - a_n\| &\leq \|a_{n+p} - l\| + \|a_n - l\| \\ &\leq J_1 \|a_N - l\| + J_2 + J_1 \|a_N - l\| + J_2 \\ &\leq J_1 \frac{\epsilon}{2J_1} - J_2 + J_2 + J_1 \frac{\epsilon}{2J_1} - J_2 + J_2 \\ &= \epsilon \end{aligned}$$

Then, (a_n) is a Cauchy sequence and converges to $c \in M$.

Lastly, to show that $c \in F(T_j, S_j)$, for any $\epsilon^* > 0$, there is N_1 such that

$$\|a_n - c\| \leq \frac{\epsilon^*}{2(2 + f_j Z)} - \frac{3g_j}{(2 + f_j Z)}, \quad \forall n \geq N_1 \tag{5}$$

Since $\lim_{n \rightarrow \infty} a_n = 0$ implies that $N_2 \geq N_1$ such that $d(a_n, F) \leq \frac{\epsilon^*}{3(4 + 3f_j Z)}, \forall n \geq N_2$.

Thus, $\exists l_1 \in F(T_j, S_j)$ such that

$$\|a_{N_2} - l_1\| \leq \frac{\epsilon^*}{2(4 + 3f_j Z)} \tag{6}$$

From (5) and (6) for any $T_j, \forall j = 1, 2, \dots, k$, we obtain

$$\begin{aligned} \|T_j c - c\| &\leq \|j c - l\| + 2\|T_j a_{N_2} - l_1\| + \|a_{N_2} - l\| \\ &\quad + \|a_{N_2} - c\| \\ &\leq \|c - l_1\| + f_j \psi \|p - l_1\| + g_j + 2\|a_{N_2} - l_1\| \\ &\quad + 2f_j \psi \|a_{N_2} - l\| + 2g_j + \|a_{N_2} - l_1\| \\ &\quad + \|a_{N_2} - c\| \\ &\leq (1 + f_j Z)\|c - l_1\| + 2(1 + f_j Z)\|a_{N_2} - l_1\| + 3g_j \\ &\quad + \|a_{N_2} - l_1\| + \|a_{N_2} - c\| \\ &\leq (1 + f_j Z)\|a_{N_2} - c\| + (1 + f_j Z)\|a_{N_2} - l_1\| \\ &\quad + 2(1 + f_j Z)\|a_{N_2} - l_1\| + 3g_j + \|a_{N_2} - l_1\| \\ &\quad + \|a_{N_2} - c\| \\ &\leq (2 + f_j Z)\|a_{N_2} - c\| + (4 + 3f_j Z)\|a_{N_2} - l_1\| \\ &\quad + 3g_j \\ &\leq (2 + f_j Z) \frac{\epsilon^*}{2(2 + f_j Z)} - 3g_j + 3g_j \\ &\quad + (4 + 3f_j Z) \frac{\epsilon^*}{2(4 + 3f_j Z)} \\ &= \epsilon^* \end{aligned}$$

Therefore, $\|T_j c - c\| = 0 \forall i$ which means $T_j c = c, \forall j = 1, 2, \dots, k$.

By using the same above argument we can prove $\|S_j c - c\| = 0, \forall j$. Thus $c \in F(T_j, S_j)$.

Theorem (2.6): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, j = 1, 2, \dots, k$ be two families of Lipschitzain and total asymptotically quasi-nonexpansive self-maps of B . If M accomplishes Opial's condition and the maps $I - T_j$ and $I - S_j, j = 1, 2, \dots, k$ are demiclosed to zero, therefore (a_n) be as shown in step (1) is weak convergence to a common fixed point of T_j and $S_j, j = 1, 2, \dots, k$.

Proof: Let $a^* \in F(T_j, S_j)$. By Lemma (2.2), $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists.

By Theorem (2.4), we have

$$\lim_{n \rightarrow \infty} \|T_j^n a_n - a_n\| = 0 = \lim_{n \rightarrow \infty} \|S_j^n a_n - a_n\|$$

for $j=1, 2, \dots, k$. Since by the supposition the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \dots, k$ are demiclosed to zero, therefore $T_j a^* = a^*$ and $S_j a^* = a^*$, that means $a^* \in F(T_j, S_j)$.

Next, to prove (a_n) converges weakly to a^* . Assume there is other subsequence (a_{n_i}) of (a_n) is weak convergence to $b^* \in F(T_j, S_j)$ and $a^* \neq b^*$. By using the same argument as above we can show that $b^* \in F(T_j, S_j)$.

Now, to prove the uniqueness, assume $a^* \neq b^*$. Therefore, by using Opial's condition, we obtain:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|a_n - a^*\| &= \lim_{n_j \rightarrow \infty} \|a_{n_j} - a^*\| \\ &< \lim_{n_j \rightarrow \infty} \|a_{n_j} - b^*\| \\ &= \lim_{n_i \rightarrow \infty} \|a_{n_i} - b^*\| \\ &< \lim_{n_i \rightarrow \infty} \|a_{n_i} - a^*\| \\ &= \lim_{n \rightarrow \infty} \|a_n - a^*\| \end{aligned}$$

This is contradiction, therefore $a^* \neq b^*$. Hence, (a_n) is weak convergence to a^* .

Theorem (2.7): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, j = 1, 2, \dots, k$ be two families of Lipschitzain and total asymptotically quasi-nonexpansive self-maps of B . If the dual space M^* of M has the Kadec-klee property and the maps $I - T_j, I - S_j, j = 1, 2, \dots, k$ are demi-closed to zero, therefore, (a_n) be as shown in step (1) is weak convergence to a common fixed point of T_j and S_j .

Proof: As showed by Lemma (2.2), that $\lim_{n \rightarrow \infty} \|a_n - a^*\|$ exists.

Since (a_n) is bounded in B and M is reflexive. Therefore, there exists a subsequence (a_{n_i}) of (a_n) which is weak convergence to a point $a^* \in B$. By Theorem (2.4)

$$\lim_{n \rightarrow \infty} \|T_j^n a_n - a_n\| = 0 = \lim_{n \rightarrow \infty} \|S_j^n a_n - a_n\|$$

$\forall j=1, 2, \dots, k$.

Since by the supposition the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \dots, k$ are demi-closed to zero. Thus, $a^* \in F(T_j, S_j)$. Now, to prove (a_n) is weak convergence to a point a^* . Presume that (a_{n_k}) is other subsequence of (a_n) that is weak convergence to a point $b^* \in F(T_j, S_j)$. By using the same argument as above, we obtain $b^* \in F(T_j, S_j)$.

Therefore, by Lemma (2.3) $\lim_{n \rightarrow \infty} \|ta_n + (1 - t)a^* - b^*\|$ exists for all $t \in [0,1]$.

By Lemma (1.8) $a^* = b^*$. As a result, the sequence (a_n) is weak convergence to the point $a^* \in F(T_j, S_j)$.

The following corollaries are special cases

Corollary (2.8): Let B be a nonempty closed convex subset of a Banach space and $T_j, S_j, j = 1, 2, \dots, k$ be two families of total asymptotically nonexpansive self-maps of B . Presume that $F(T_j, S_j) \neq \emptyset$ and $\sum_{n=1}^{\infty} f_n < \infty, \sum_{n=1}^{\infty} g_n < \infty$.

Presume that (a_n) be as shown in step (1) is strong convergence to a common fixed point of T_j and S_j iff $\lim_{n \rightarrow \infty} \inf d(a_n, F) = 0$, where $d(a, F) = \inf_{a^* \in F} \|a - a^*\|$.

Corollary (2.9): Let T_j, S_j, B, f_j and $g_j, j = 1, 2, \dots, k$ be as in corollary (2.8). Therefore (a_n) be as shown in step (1) is strong convergence to $a^* \in F(T_j, S_j)$ iff (a_{n_i}) of (a_n) that converges to a^* .

Corollary (2.10): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \dots, k$ be two families of Lipschitzain and total asymptotically nonexpansive self-maps of B . If the dual space M^* of M has the Kadec-klee property and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \dots, k$ are demi-closed to zero, therefore (a_n) be as shown in step (1) is weak convergence to a common fixed point of T_i and S_i .

Corollary (2.11): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \dots, k$ be two families of total asymptotically nonexpansive self-maps of B . If M accomplishes Opial's condition and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \dots, k$ are demi-closed to zero, therefore, (a_n) be as shown in step (1) is weak convergence to a common fixed point of $T_j, S_j, \forall j = 1, 2, \dots, k$.

Corollary (2.12): Let B be a nonempty closed convex subset of a Banach space and $T_j, S_j, j = 1, 2, \dots, k$ be two families of asymptotically quasi-nonexpansive self-maps of B . Presume that $F(T_j, S_j) \neq \emptyset$ and $\sum_{n=1}^{\infty} f_n < \infty$. Presume that (a_n) be as shown in step (1) is strong convergence to a common fixed point of T_j and S_j iff $\lim_{n \rightarrow \infty} \inf d(a_n, F) = 0$, where $d(a, F) = \inf_{a^* \in F} \|a - a^*\|$.

Corollary (2.13): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \dots, k$ be two families of Lipschitzain and asymptotically quasi-nonexpansive self-maps of B . If the dual space M^* of M has the Kadec-klee property and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \dots, k$ are demi-closed to zero, therefore, (a_n) be as shown in step (1) is weak convergence to a common fixed point of T_j and S_j .

Corollary (2.14): Let B be a nonempty closed convex subset of a uniformly convex Banach space and $T_j, S_j, \forall j = 1, 2, \dots, k$ be two families of Lipschitzain and asymptotically quasi-nonexpansive selfmaps of B . If M accomplishes Opial's condition and the maps $I - T_j$ and $I - S_j, \forall j = 1, 2, \dots, k$ are demi-closed to zero, therefore, (a_n) be as shown in step (1) is weak convergence to a common fixed point of T_j and $S_j, \forall j = 1, 2, \dots, k$.

3. Numerical Example

We illustrate our results by the following

Example (3.1): Let $T_j, S_j: R \rightarrow R, \forall j = 1, 2, \dots, k$ be two maps such that $T_j a = \frac{2a}{3^j}$ and $S_j a = \frac{46a}{2^j}, \forall a \in R$. Choose $\alpha_{jn} = \frac{n}{4^{(n+1)}} \forall n$ with initial value $a_1 = 15$. Let (a_n) be the sequence. According to Table 1 and Figure 1, it eases to sight that (a_n) converges to the fixed point $a^* = 0$.

Table 1: Numerical results corresponding to $a_1 = 15$ for 36 steps.

n	Iteration (1)	n	Iteration (1)
1	15.0000	13	0.0184
2	9.4401	14	0.0102
3	5.6594	15	0.0056
4	3.3084	16	0.0031
5	1.9044	17	0.0017
6	1.0849	18	0.0009
7	0.6134	19	0.0005
8	0.3448	20	0.0003
9	0.1930	21	0.0002
10	0.1076	22	0.0001
11	0.0599	23	0.0000
12	0.0332	24	0.0000

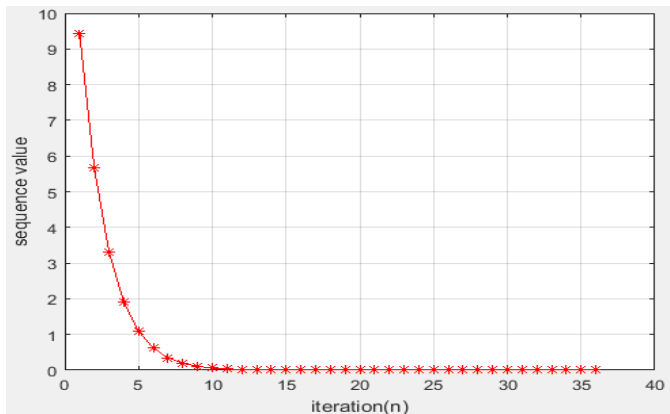


Figure. 1. Convergence behavior corresponding to $a_1 = 15$ for 36 steps.

4. Conclusion

We study the strongly and weakly convergence of new type of finite-step iteration processes under total asymptotically quasi-nonexpansive maps, see Theorems (2.4)-(2.6). Our results are generalizing and unifying the results of others who have been referred to in the references.

5. Open Problem

Recently, S.S. Abed has been defined as the following type of generalizations of total asymptotically quasi-nonexpansive[16]: Let \mathcal{A} be a subset of real Banach space a set-valued map $G: \mathcal{A} \rightarrow 2^{\mathcal{A}}$ is called the general asymptotic set-valued if for each $x \in \mathcal{A}$ there exists null non- negative real sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\mathfrak{D}(G^n x, G^n y) \leq \|x - w\| + a_n \mu(\|x - w\|) + b_n \quad (7)$$

for any $y \in \mathcal{A}$, $w \in G^n y$ and $\mu: R^+ \rightarrow R^+$ with $\mu(0) = 0$. One can study convergence theorems in (1) and in [17, theorem (11)] for families of general asymptotic set-valued maps. As well as possible to demonstrate new results in the case of other spaces as a modular space [18].

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