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L^{∞}-Estimates for Nonlinear Degenerate Elliptic Problems with p-growth in the Gradient

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A B S T R A C T

In this work, we will prove the existence of bounded solutions for the nonlinear elliptic equations

$$-\,div(a(x,u,\nabla u)) = g(x,u,\nabla u) - \,div f,$$

in the setting of the weighted Sobolev space $W^{1,p}(\Omega,w)$ where a, g are Carathéodory functions which satisfy some conditions and f satisfies suitable summability assumption.

1 Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^N$, $N > 1$ and let us consider the problem:

$$\begin{cases} -\,\mathrm{div}\Big(a(x,u,\nabla u)\Big) = g(x,u,\nabla u) - \,\mathrm{div}\, f \text{ in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega,w)\cap L^{\infty}(\Omega), \end{cases} \quad (1)$$

where $-\mathrm{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^{1-p'})$ with $p > 1$ and $\frac{1}{p}+\frac{1}{p'} = 1$, g is a nonlinearity which satisfies the growth condition, but it does not satisfy any sign condition. And f satisfies suitable summability assumption.

In [1], the authors proved the existence results in the setting of weighted Sobolev spaces for quasilinear degenerated elliptic problems associated with the following equation $-\,\mathrm{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = f$, where g satisfies the sign condition.

In [2], Benkirane and Bennouna studied L^{∞} estimates of the solutions in $W_0^{1,p}(\Omega,w)$ of the problem $-\,\mathrm{div}\, a(x,u,\nabla u) - \,\mathrm{div}\, \phi(u) + g(x,u) = f$ with a non-uniform elliptic condition, and g satisfies the sign condition.

In [3], the authors proved the existence of bounded solutions of the problem $-\,\mathrm{div}\, a(x,u,\nabla u) = g - \,\mathrm{div}\, f$, whose principal part has a degenerate coercivity, in the setting of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$.

The equations like (1) have been studied by many authors in the non-degenerate case (i.e. with $w(x)\equiv 1$) (see, e.g., [4] and the references therein).

The aim of this article is to establish a bounded solution for the problem (1) based on rearrangement properties. The results of this work can be considered as an extension of the results in [4] to the weighted case.

In order to perform L^{∞}-Estimates, the paper is organized in the following way. In section 1, we presented the introduction of the current work. In Section 2 we will state some basic knowledge of Sobolev spaces with weight and properties of the relative rearrangement. Finally in Section 3, we will introduce the essential assumptions, and we will prove our main result.

2 Preliminary results

2.1 Sobolev spaces with weight

In order to discuss the problem (1), we need some theories on $W^{1,p}(\Omega,w)$ which is called Sobolev spaces with weight.

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Firstly we state some basic properties of spaces $W^{1,p}(\Omega,w)$ which will be used later (for details, see [5]). Let Ω be an open subset of $\mathbb{R}^N$ with $N \geq 2$, and $1 \leq p < \infty$ a real number.

Let $w = w(x)$ be a weighted function which is measurable and positive function a.e. in Ω. Define $L^p(\Omega,w) = \{u \ measurable : uw^{\frac{1}{p}} \in L^p(\Omega)\}$. We shall denote by $W^{1,p}(\Omega,w)$ the function space which consists of all real functions $u \in L^p(\Omega)$ such that their weak derivatives $\frac{\partial u}{\partial x_i}$, for all $i = 1,\dots,N$ (in the sense of distributions) satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega,w)$, for all $i = 1,\dots,N$. Endowed with the norm

$$|||u|||_{p,w} = \left(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p w(x) dx\right)^{\frac{1}{p}}, \quad (2)$$

$W^{1,p}(\Omega,w)$ is a Banach space. Further more, we suppose that

$$w \in L^1_{loc}(\Omega), \quad (3)$$

$$w^{-\frac{1}{p-1}} \in L^1(\Omega), \quad (4)$$

Due to condition (3), $\mathcal{C}_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega,w)$.

Since we are dealing with compactness methods to get solutions of nonlinear elliptic equations, a compact imbedding is necessary. This leads us to suppose that the weight function w also satisfies

$$w^{-q} \in L^1(\Omega), \text{ where } 1 + \frac{1}{q} < p \text{ and } q > \frac{N}{p}. \quad (5)$$

Condition (5) ensures that the imbedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow L^p(\Omega) \quad (6)$$

is compact.

Therefore, we denote by $W_0^{1,p}(\Omega,w)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p,w} = \left(\int_\Omega |\nabla u|^p w(x) dx\right)^{\frac{1}{p}}.$$

We remark that condition (4) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces if $1 < p < \infty$.

Let us give the following lemmas which will be needed later.

Lemma 2.1 *(See [1]) Assume that* (6) *holds. Let $F : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function such that $F(0) = 0$. Then, F maps $W_0^{1,p}(\Omega,w)$ into itself. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(Fou)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2 *(See [1]) Let $u \in L^r(\Omega,w)$ and let $u_n \in L^r(\Omega,w)$, with $\|u_n\|_{L^r(\Omega,w)} \leq c$, $1 < r < \infty$. If $u_n \to u$ a.e. in Ω, then $u_n \rightharpoonup u$ in $L^r(\Omega,w)$, where $\rightharpoonup$ denotes weak convergence.*

2.2 Properties of the relative rearrangement

In this paragraph, we recall some standard notations and properties about decreasing rearrangements which will be used throughout this paper.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $v : \Omega \to \mathbb{R}$ be a measurable function. If one denotes by $|E|$ the Lebesgue measure of a set E, one can define the distribution function $\mu_v(t)$ of v as:

$$\mu_v(t) = |\{x \in \Omega : |v(x)| > t\}|, \ t \geq 0.$$

The decreasing rearrangement v^* of v is defined as the generalized inverse function of μ_v:

$$v^*(s) = \ \inf\{t \geq 0 : \mu_v(t) \leq s\}, \ s \in [0,|\Omega|].$$

We recall that v and v^* are equimeasurable, i.e.,

$$\mu_v(t) = \mu_{v^*}(t), \ t \in \mathbb{R}^+.$$

This implies that for any Borel function ψ, it holds that

$$\int_\Omega \psi(v(x))dx = \int_0^{|\Omega|} \psi(v^*(s))ds.$$

In particular,

$$\|v^*\|_{L^p(0,|\Omega|)} = \|v\|_{L^p(\Omega)}, \ 1 \leq p < \infty. \quad (7)$$

and, if $v \in L^\infty(\Omega)$,

$$v^*(0) = ess\sup_\Omega |v|.$$

The theory of rearrangements is well known, and its exhaustive treatments can be found for example in [6, 7,8]. Now we recall two notions which allow us to define a "generalized" concept of rearrangement of a function f with respect to a given function v.

Definition 2.1 *(See [9]). Let $f \in L^1(\Omega)$ and $v \in L^1(\Omega)$. We will say that a function $\overline{f}_v \in L^1(0,|\Omega|)$ is a pseudo-rearrangement of f with respect to v if there exists a family $\{D(s)\}_{s\in(0,|\Omega|)}$ of subsets of Ω satisfying the properties:*

(i) $|D(s)| = s$,

(ii) $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$,

(iii) $D(s) = \{x \in \Omega : v(x) > t\}$ *if* $s = \mu_v(t)$,

such that

$$\overline{f}_v(s) = \frac{d}{ds}\int_{D(s)} f(x), \text{ in } \mathcal{D}'(\Omega).$$

Definition 2.2 *(See [10]). Let $f \in L^1(\Omega)$ and $v \in L^1(\Omega)$. The following limit exists:*

$$\lim_{\lambda \searrow 0} \frac{(v+\lambda f)^* - v^*}{\lambda} = f_v^*,$$

where the convergence is in $L^p(\Omega)$-weak, if $f \in L^p(\Omega), 1 \leq p < \infty$, and in $L^\infty(\Omega)-weak^*$, if $f \in L^\infty(\Omega)$. The function

f_v^* is called the relative rearrangement of f with respect to v. Moreover, one has

$$f_v^*(s) = \frac{dG}{ds}, \ in \ \mathcal{D}'(\Omega),$$

where

$$G(s) = \int_{\{v>v^*(s)\}} f(x)dx + \int_0^{s-|\{v>v^*(s)\}|} (f|_{\{v=v^*(s)\}})(\sigma)d\sigma.$$

The two notions are equivalent in some precise sense (see [6]). For this reason we will denote both $\overline{f}_v$ and f_v^* by F_v. We only recall a few results which hold for both the pseudo- and the relative rearrangements.

If f and v are non-negative and $v \in W_0^{1,1}(\Omega)$, it is possible to prove the following properties:

$$-\frac{d}{dt}\int_{\{v>t\}} f(x)dx = F_v(\mu_v(t))(-\mu_v'(t)), \text{ for a.e. } t>0; \quad (8)$$

$$\|F_v\|_{L^p(0,|\Omega|)} \le \|f\|_{L^p(\Omega)}, \quad 1 \le p < \infty. \quad (9)$$

The proofs of (8) and (9) can be found in [9] (for pseudo-rearrangements) and in [11, 12] (for relative rearrangements). We finally recall the following chain of inequalities which holds for any non-negative $v \in W_0^{1,p}(\Omega)$:

$$\begin{aligned} N C_N^{1/N} \mu_v(t)^{1-1/N} &\le -\frac{d}{dt}\int_{\{v>t\}} |\nabla v| dx \\ &\le (-\mu_v'(t))^{1/p'} \left(-\frac{d}{dt}\int_{\{v>t\}} |\nabla v|^p dx\right)^{1/p}, \end{aligned} \quad (10)$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$. It is a consequence of the Fleming-Rishel formula [13], the isoperimetric inequality [14] and the Hölder's inequality.

3 Assumptions and main results

Let us now give the precise hypotheses on the problem (1), we assume that the following assumptions: Ω is a bounded open set of $\mathbb{R}^N$ ($N > 1$), $1 < p < +\infty$, Let w be a non-negative real valued measurable function defined on Ω which satisfies (3), (4) and (5). Let $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, such that

$$a(x,s,\xi)\xi \ge \alpha w(x)|\xi|^p, \quad (11)$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \quad (12)$$

where α is a strictly positive constant and for all $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$, with $\xi \neq \eta$.

$$|a(x,s,\xi)| \le \beta w(x)^{\frac{1}{p}}(d(x) + |s|^{p-1} + w(x)^{\frac{1}{p'}}|\xi|^{p-1}), \quad (13)$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, some positive function $d(x) \in L^{p'}(\Omega)$, $1 < p \le N$, and $\beta > 0$.

Furthermore, let $g(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies, for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the following condition

$$|g(x,s,\xi)| \le b_1(x) + b_2(x)w(x)|\xi|^p, \quad (14)$$

where $b_1(x) \in L^m(\Omega)$, $\frac{1}{m} < \frac{p}{N} - \frac{1}{q}$, $m > 1$ and $b_1(x) \ge 0$ a.e. $|b_2(x)| \le \lambda$ a.e. in Ω where λ is a strictly positive constant.

Finally, the right hand side we assume that

$$f w^{-\frac{1}{p}} \in (L^{mp'}(\Omega))^N. \quad (15)$$

Now, we give the definition of weak solutions of problem (1).

Definition 3.1 *We say that a function $u \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$ is a weak solution of problem (1) if*

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v\, dx = \int_\Omega g(x,u,\nabla u)v\, dx + \int_\Omega f\cdot\nabla v\, dx, \quad (16)$$

for all $v \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$.

Our main results are collected in the following existence result:

Theorem 3.1 *Suppose that the assumptions (3)–(5) and (11)–(15) hold true. Then there exists at least one weak solution $u \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$ of problem (1).*

And in the following theorem

Theorem 3.2 *Let u be a solution of (1) and let us assume that (3)–(5) and (11)–(15) hold true. If b_1 and f satisfy the inequality*

$$\begin{aligned} &\left(N C_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \\ &\times \|w^{-q}\|^{\frac{1}{q(p-1)}}_{L^1(\Omega)} \left(\frac{N\gamma}{q} \times \frac{q(p-1)-1}{p\gamma - N}\right)^{1-\frac{1}{q(p-1)}} |\Omega|^{\frac{p\gamma-N}{N\gamma(p-1)}} \\ &+ \frac{p'}{\alpha^{p'/p} N C_N^{1/N}} \|w^{-q}\|^{\frac{1}{qp}}_{L^1(\Omega)} \|f\|^{\frac{1}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N} \times \\ &\times \left(\frac{N(p\gamma-1)}{p\gamma-N}\right)^{1-\frac{1}{p\gamma}} |\Omega|^{\frac{p\gamma-N}{Np\gamma}} \\ &\le \frac{\alpha(p-1)}{\lambda p'}, \end{aligned} \quad (17)$$

where $1/\gamma = 1/m + 1/q$.
Then there exists a constant $M > 0$, which depends only on N, p, p', q, $|\Omega|$, $\|f\|_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}$, $\|w^{-q}\|_{L^1(\Omega)}$ and $\|b_1\|_{L^m(\Omega)}$, such that

$$\|u\|_{L^\infty(\Omega)} \le M. \quad (18)$$

Lemma 3.1 *Let u be a solution of (1) and let us assume that (3)–(5) and (11)–(15) hold true. Define*

$$\varphi = \frac{e^{k|u|}-1}{k}, \qquad k = \frac{\lambda p'}{\alpha(p-1)}. \quad (19)$$

Then the decreasing rearrangement of φ satisfies the following differential inequality:

$$\begin{aligned}
&(-\varphi^*(s))' \\
&\leq \frac{\left[\left(w(x)^{-\frac{1}{p-1}}\right)^*_\varphi\right]^{1/p'}(s)[(-\varphi^*(s))']^{1/p}}{NC_N^{1/N}s^{1-1/N}}\times \\
&\quad\times\left(\int_0^s \psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\right)^{1/p} \\
&+\frac{k\varphi^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\left[\left(w(x)^{-\frac{1}{p-1}}\right)^*_\varphi\right]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{1/p}
\end{aligned} \tag{20}$$

Proof. Let us define two real functions $\phi_1(z)$, $\phi_2(z)$, $z\in\mathbb{R}$, as follows:

$$\begin{cases} \phi_1(z) = e^{k(p-1)|z|}\,\mathrm{sign}(z), \\ \phi_2(z) = \frac{(e^{kz}-1)}{k}, \end{cases} \tag{21}$$

where $k = \frac{\lambda p'}{\alpha(p-1)}$, we observe that $\phi_2(0) = 0$ and for $z \neq 0$, $\phi_1'(z) > 0$, $\phi_2'(z) > 0$,

$$\phi_1(z)\phi_2'(|z|)\,\mathrm{sign}(z) = |\phi_2'(|z|)|^p, \tag{22}$$

$$\phi_1'(z) - \frac{\lambda p'}{\alpha}|\phi_1(z)| = 0. \tag{23}$$

Furthermore, for $t > 0$, $h > 0$, let us put

$$S_{t,h}(z) = \begin{cases} \mathrm{sign}(z) & \text{if } |z| > t+h, \\ ((|z|-t)/h)\,\mathrm{sign}(z) & \text{if } t < |z| \leq t+h, \\ 0 & \text{if } |z| \leq t. \end{cases} \tag{24}$$

We use in (1) the test function $v \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$ defined by

$$v = \phi_1(u)S_{t,h}(\varphi) = \phi_1(u)S_{t,h}(\phi_2(|u|)), \tag{25}$$

where $\varphi = \frac{e^{k|u|}-1}{k}$. Using (24) we have

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} a(x,u,\nabla u)\nabla u\phi_1(u)\phi_2'(|u|)\mathrm{sign}\,(u)\,dx \\
&-\int_{\{\varphi>t\}} g(x,u,\nabla u)\phi_1(u)S_{t,h}(\varphi)\,dx \\
&+\int_{\{\varphi>t\}} a(x,u,\nabla u)\nabla u\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&=\int_{\{\varphi>t\}}\sum_{i=1}^N f_i\frac{\partial u}{\partial x_i}\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&+\frac{1}{h}\int_{\{t<\varphi\leq t+h\}}\sum_{i=1}^N f_i\frac{\partial u}{\partial x_i}\phi_1(u)\phi_2'(|u|)\mathrm{sign}\,(u)\,dx.
\end{aligned} \tag{26}$$

Taking into account (22) and Young's inequality, it follows that

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} a(x,u,\nabla u)\nabla u\phi_1(u)\phi_2'(|u|)\mathrm{sign}(u)\,dx \\
&-\int_{\{\varphi>t\}} g(x,u,\nabla u)\phi_1(u)S_{t,h}(\varphi)\,dx \\
&+\int_{\{\varphi>t\}} a(x,u,\nabla u)\nabla u\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&\leq\frac{\alpha^{-p'/p}}{p'}\int_{\{\varphi>t\}}|f|^{p'}w^{\frac{-p'}{p}}\phi_1'(u)S_{t,h}(\varphi)dx \\
&+\frac{\alpha}{p}\int_{\{\varphi>t\}} w(x)|\nabla u|^p\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&+\frac{\alpha^{-p'/p}}{p'h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}|\phi_2'(|u|)|^p\,dx \\
&+\frac{\alpha}{ph}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx,
\end{aligned}$$

using (14), and the ellipticity condition (11), we obtain

$$\begin{aligned}
&\frac{\alpha}{h}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx \\
&\leq\int_{\{\varphi>t\}}\Big(\Big(b_1(x)+b_2(x)w(x)|\nabla u|^p\Big)\phi_1(u) \\
&-\alpha w(x)|\nabla u|^p\phi_1'(u)\Big)S_{t,h}(\varphi)\,dx \\
&+\frac{\alpha^{-p'/p}}{p'}\int_{\{\varphi>t\}}|f|^{p'}w^{\frac{-p'}{p}}\phi_1'(u)S_{t,h}(\varphi)dx \\
&+\frac{\alpha}{p}\int_{\{\varphi>t\}} w(x)|\nabla u|^p\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&+\frac{\alpha^{-p'/p}}{p'h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}|\phi_2'(|u|)|^p\,dx \\
&+\frac{\alpha}{ph}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx,
\end{aligned}$$

By(23), it follows that

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx \\
&\leq\int_{\{\varphi>t\}}\left(\frac{\lambda p'}{\alpha}|\phi_1(u)|-\phi_1'(u)\right)w(x)|\nabla u|^pS_{t,h}(\varphi)\,dx \\
&+\int_{\{\varphi>t\}}\left(\frac{p'}{\alpha}b_1(x)+\frac{\lambda p'}{\alpha^{p'+1}}|f|^{p'}w^{\frac{-p'}{p}}\right)|\phi_1(u)|S_{t,h}(\varphi)\,dx \\
&+\frac{1}{\alpha^{p'}h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}|\phi_2'(|u|)|^p\,dx.
\end{aligned}$$

Using (23) and the definition of ϕ_1, ϕ_2 in (21), the above inequality gives:

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla\varphi|^p dx \\
&\qquad\leq\int_{\{\varphi>t\}}\psi(k\varphi+1)^{p-1}S_{t,h}(\varphi)dx \\
&\qquad+\frac{1}{\alpha^{p'}h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}(k\varphi+1)^p dx
\end{aligned} \tag{27}$$

where $\psi = \frac{p'}{\alpha}b_1(x)+\frac{\lambda p'}{\alpha^{p'+1}}|f|^{p'}w^{\frac{-p'}{p}}$.

Letting h go to 0 in a standard way we get:

$$-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)|\nabla\varphi|^p\,dx \le \int_{\{\varphi>t\}} \psi(k\varphi+1)^{p-1}\,dx + \frac{(kt+1)^p}{\alpha^{p'}}\Big(-\frac{d}{dt}\int_{\{\varphi>t\}} F(x)\,dx\Big),$$

with $F(x)=|f(x)|^{p'}w^{\frac{-p'}{p}}(x)$.
Using Hardy-Littlewood's inequality and the inequality (8). It follows that

$$\begin{aligned}-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)|\nabla\varphi|^p\,dx &\le \int_0^{\mu_\varphi(t)} \psi^*(s)(k\varphi^*(s)+1)^{p-1}\,ds \\ &+\frac{(kt+1)^p}{\alpha^{p'}}(-\mu'_\varphi(t))F_\varphi(\mu_\varphi(t)),\end{aligned} \tag{28}$$

where F_φ is a pseudo-rearrangement (or the relative rearrangement) of $|f|^{p'}$ with respect to φ.

On the other hand, thanks to Hölder's inequality, we can easily check that

$$\begin{aligned}-\frac{d}{dt}\int_{\{\varphi>t\}}|\nabla u|\,dx &\le \Big(-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)|\nabla u|^p\,dx\Big)^{\frac{1}{p}}\times \\ &\times\Big(-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)^{-\frac{1}{p-1}}\,dx\Big)^{1-\frac{1}{p}}.\end{aligned} \tag{29}$$

Since $w(x)^{-\frac{1}{p-1}}\in L^1(\Omega)$, we write

$$-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)^{-\frac{1}{p-1}}\,dx = \Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi(\mu_\varphi(t))\times(-\mu'_\varphi(t)), \tag{30}$$

for almost every $t>0$, where $\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi$ is the relative rearrangement of $w(x)^{-\frac{1}{p-1}}$ with respect to φ.
Using the Fleming–Rishel formula (see [8]), we can write

$$-\frac{d}{dt}\int_{\{\varphi>t\}}|\nabla u|\,dx \ge NC_N^{\frac{1}{N}}(\mu_\varphi(t))^{1-\frac{1}{N}} \tag{31}$$

for almost every $t>0$.
Combining (28), (29), (30) and (31), we obtain

$$\begin{aligned}&NC_N^{1/N}\mu_\varphi(t)^{1-1/N} \\ &\le \Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(\mu_\varphi(t))(-\mu'_\varphi(t))^{1/p'}\times \\ &\quad\times\Big(\int_0^{\mu_\varphi(t)}\psi^*(s)(k\varphi^*(s)+1)^{p-1}ds\Big)^{1/p} \\ &+\frac{kt+1}{\alpha^{p'/p}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(\mu_\varphi(t))(-\mu'_\varphi(t))\Big(F_\varphi(\mu_\varphi(t))\Big)^{1/p}\end{aligned}$$

and then, using the definition of $\varphi^*(s)$, we have:

$$\begin{aligned}&(-\varphi^*(s))' \\ &\le \frac{\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)[(-\varphi^*(s))']^{1/p}}{NC_N^{1/N}s^{1-1/N}}\times \\ &\quad\times\Big(\int_0^s\psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\Big)^{1/p} \\ &+\frac{k\varphi^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{1/p}\end{aligned}$$

that is (20).

Proof of Theorem 3.2. By using Young's inequality and (20) of Lemma 3.1 implies:

$$\begin{aligned}(-\varphi^*(s))' &\le \frac{1}{p}(-\varphi^*(s))' + \frac{\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)}{p'(NC_N^{1/N}s^{1-1/N})^{p'}}\times \\ &\times\Big(\int_0^s\psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\Big)^{p'/p} \\ &+\frac{k\varphi^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}.\end{aligned}$$

We deduce that

$$\begin{aligned}(-\varphi^*(s))' &\le \frac{\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)}{(NC_N^{1/N}s^{1-1/N})^{p'}}\times \\ &\times\Big(\int_0^s\psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\Big)^{p'/p} \\ &+\frac{p'(k\varphi^*(s)+1)}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}.\end{aligned}$$

Integrating between 0 and $|\Omega|$, since $\varphi(|\Omega|)=0$, we have

$$\begin{aligned}\varphi^*(0) &= \int_0^{|\Omega|}(-\varphi^*(s))'ds \\ &\le \int_0^{|\Omega|}\frac{1}{(NC_N^{1/N}s^{1-1/N})^{p'}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)\Big(\int_0^s\psi^*(\tau)d\tau\Big)^{p'/p}ds\times \\ &\times(k\varphi^*(0)+1)^{(p-1)p'/p} \\ &+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}ds\times \\ &\times(k\varphi^*(0)+1).\end{aligned}$$

Since φ^* attains its maximum at 0, we can write

$$\|\varphi\|_{L^\infty(\Omega)} \le kA\|\varphi\|_{L^\infty(\Omega)} + A, \tag{32}$$

where $k=\frac{\lambda p'}{\alpha(p-1)}$ and

$$\begin{aligned}A &= \int_0^{|\Omega|}\frac{1}{(NC_N^{1/N}s^{1-1/N})^{p'}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)\Big(\int_0^s\psi^*(\tau)d\tau\Big)^{p'/p}ds \\ &+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}ds \\ &= I+J.\end{aligned}$$

In order to estimate I, we use the Hölder's inequality and (7), obtaining

$$\begin{aligned}\int_0^s \psi^*(\tau)d\tau &\leq \|\psi\|_{L^m(\Omega)} s^{1-1/m} \\ &\leq (\tfrac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \tfrac{\lambda p'}{\alpha^{p'+1}}\||f w^{\frac{-1}{p}}|^{p'}\|_{L^m(\Omega)}) s^{1-1/m} \\ &\leq (\tfrac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \tfrac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}) s^{1-1/m},\end{aligned} \tag{33}$$

which we use to get

$$I \leq \left(NC_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \int_0^{|\Omega|} \left(w(x)^{-\frac{1}{p-1}}\right)^*_{\varphi}(s) s^{\frac{p'}{p}(1-\frac{1}{m})-p'(1-\frac{1}{N})} ds.$$

By (5) one has $q(p-1) > 1$, we use again Hölder's inequality to obtain

$$I \leq \left(NC_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \|w^{-q}\|^{\frac{1}{q(p-1)}}_{L^1(\Omega)} \left(\int_0^{|\Omega|} t^{(\frac{p'}{p}(1-\frac{1}{m})-p'(1-\frac{1}{N}))\frac{q(p-1)}{q(p-1)-1}} dt\right)^{1-\frac{1}{q(p-1)}}.$$

The assumptions on exponents (5) and (14) allow us to get

$$I \leq \left(NC_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \|w^{-q}\|^{\frac{1}{q(p-1)}}_{L^1(\Omega)} \left(\frac{N\gamma}{q} \times \frac{q(p-1)-1}{p\gamma - N}\right)^{1-\frac{1}{q(p-1)}} |\Omega|^{\frac{p\gamma-N}{N\gamma(p-1)}}. \tag{34}$$

We now turn to estimate J. Since $p\gamma > N > 1$, we can consider the Hölder conjugate exponent $\eta = \frac{p\gamma}{p\gamma-1}$. The conjugate exponent η satisfies the identity

$$\frac{1}{qp} + \frac{1}{mp} + \frac{1}{\eta} = 1,$$

so that by Hölder's inequality we obtain

$$J \leq \frac{p'}{\alpha^{p'/p} NC_N^{1/N}} \|w^{-q}\|^{\frac{1}{qp}}_{L^1(\Omega)} \left(\int_\Omega |f|^{mp'} w^{\frac{-m}{p-1}} dx\right)^{\frac{1}{mp}} \times \left(\int_0^{|\Omega|} s^{\eta(\frac{1}{N}-1)} ds\right)^{\frac{1}{\eta}}.$$

Then we have

$$J \leq \frac{p'}{\alpha^{p'/p} NC_N^{1/N}} \|w^{-q}\|^{\frac{1}{qp}}_{L^1(\Omega)} \|f\|^{\frac{1}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N} \times \left(\frac{N(p\gamma-1)}{p\gamma-N}\right)^{1-\frac{1}{p\gamma}} |\Omega|^{\frac{p\gamma-N}{Np\gamma}}. \tag{35}$$

Using (34) and (35) we can estimate the quantity A in (32), Obtaining that under assumption (17) the following inequality holds:

$$\frac{\lambda p'}{\alpha(p-1)} A < 1.$$

Then (32) implies (18).

Proof of Theorem 3.1.

Let us define for $\varepsilon > 0$ the approximation

$$g_\varepsilon(x,s,\xi) = \frac{g(x,s,\xi)}{1+\varepsilon|g(x,s,\xi)|}.$$

On note that $|g_\varepsilon(x,s,\xi)| \leq |g(x,s,\xi)|$, and $|g_\varepsilon(x,s,\xi)| < \frac{1}{\varepsilon}$. we consider the approximate problem

$$\begin{cases} -\operatorname{div}\Big(a(x,u_\varepsilon,\nabla u_\varepsilon)\Big) = g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) - \operatorname{div} f \text{ in } \mathcal{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega). \end{cases} \tag{36}$$

Since the operator $A - g_\varepsilon : W_0^{1,p}(\Omega,w) \to W^{-1,p'}(\Omega,w^{1-p'})$ is bounded, coercive, and pseudo-monotone operator, where $A(u) = -\operatorname{div}\Big(a(x,u,\nabla u)\Big)$, there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega,w)$ of the problems (36)(see [15], [16] and in the weighted case [1]).
Using Stampacchia's method [17], one can prove that any solution u_ε of (36) belongs to $L^\infty(\Omega)$ for fixed ε. Finally using the L^∞ estimate obtained in Theorem 3.2 and working as in [15] and [1], but with obvious modifications, we obtain Theorem 3.1.

4 References

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