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Nonlinear parabolic problem with lower order terms in Musielak-Orlicz spaces

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ABSTRACT

We prove an existence result of entropy solutions for the nonlinear parabolic problems: $\frac{\partial b(x,u)}{\partial t}+A(u)-div(\Phi(x,t,u))+H(x,t,u,\nabla u)=f$, *and* $A(u)=-div(a(x,t,u,\nabla u))$ *is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz space, the term* $\Phi(x,t,u)$ *is a Crathéodory function assumed to be continuous on u and satisfy only the growth condition* $\Phi(x,t,u)\leq c(x,t)\overline{M}^{-1}M(x,\alpha_0 u)$, *prescribed by Musielak-Orlicz functions M and* $\overline{M}$ *which inhomogeneous and not satisfy* Δ_2*-condition,* $H(x,t,u,\nabla u)$ *is a Crathéodory function not satisfies neither the sign condition or coercivity and* $f\in L^1(Q_T)$.

1 Introduction

Let Ω be a bounded open set of $I\!R^N$ $(N\geq 2)$, T is a positive real number, and $Q_T=\Omega\times(0,T)$. Consider the following nonlinear Dirichlet equation:

$$\begin{cases} \frac{\partial b(x,u)}{\partial t}+A(u)-div(\Phi(x,t,u))+H(x,t,u,\nabla u)=f,\\ u(x,t)=0 \quad on \quad \partial\Omega\times(0,T),\\ b(x,u)(t=0)=b(x,u_0) \quad in \quad \Omega. \end{cases} \tag{1}$$

where $A(u)=-div(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is a Musielak-Orlicz-function related to the growths of the Carathéodory functions $a(x,t,u,\nabla u)$, $\Phi(x,t,u)$ and $H(x,t,u,\nabla u)$ (see assumptions (12), (15) and (16). $b:\Omega\times I\!R\to I\!R$ is a Carathéodory function such that for every $x\in\Omega$, $b(x,.)$ is a strictly increasing $\mathcal{C}^1(I\!R)$-function, the data f and $b(.,u_0)$ in $L^1(Q_T)$ and $L^1(\Omega)$ respectively.
Starting with the prototype equation:

$$\frac{\partial u}{\partial t}-\triangle_p(u)+div(c(.,t)|u|^{\gamma-1}u)+b|\nabla u|^{\delta}=f, \text{ in } Q_T.$$

In the Classical Sobolev-spaces, the authors in [1] have proved the existence of weak solutions, with $c(.,.)\equiv 0$. For $c(.,.)\in L^2(Q_T)$ and $p=2$, in [2] have proved the existence of entropy solutions, recently in [3] have proved an existence results of renormalized solutions in the case where $p\geq 2$ and $c(.,.)\in L^r(Q_T)$ with $r>\frac{N+p}{p-1}$, and by in [4] for more general parabolic term. For the elliptic version of the problem (1), more results are obtained see e.g. [5-7].

In the degenerate Sobolev-spaces an existence results is shown in [8] without sign condition in $H(x,t,u,\nabla u)$.

In the Orlicz-Sobolev spaces, the existence of entropy solutions of the problem (1) in [9] is proved where $H(x,t,u,\nabla u)\equiv 0$ and the growth of the first lower order Φ prescribed by an isotropic N-function P with $(P<<M)$. To our knowledge, differential equations in general MusielakSobolev spaces have been studied rarely see [10-14], then our aim in this paper is to overcome some difficulties encountered in these spaces and to generalize the result of [4, 9, 15, 16], and we prove an existence result of entropy solution for the obstacle parabolic problem (1), with less restrictive growth, and no coercivity condition in the first lower order term Φ, and without sign condition in the second lower order H, in the framework of inhomogeneous Orlicz-Sobolev spaces $W_0^{1,x}L_M(Q_T)$, and N-function M, defining space does not satisfy the $\triangle_2$-condition.

This paper is organized as follows. In section 2, we recall some definitions, properties and technical lemmas about Musielak Orlicz Sobolev , In section 3 is devoted to specify the assumptions on b,Φ,f,u_0,

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giving the definition of a entropy solution of (1) and we establish the existence of such a solution Theorem 4. In section 4, we give the proof of Theorem 4.

2 Musielak-Orlicz space and a technical lemma

In this part we will define the musielak-Orlicz function which control the growth of our operator.

2.1 Musielak-Orlicz function

Let Ω be an open subset of $I\!R^N$ ($N \geq 2$), and let M be a real-valued function defined in $\Omega \times I\!R_+$ and satisfying conditions:
(Φ_1):$M(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $M(x,0) = 0$,
$M(x,0) > 0$ for $t > 0$, $\lim_{t\to 0} \sup_{x\in\Omega} \frac{M(x,t)}{t} = 0$ and $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{M(x,t)}{t} = \infty$).
(Φ_2):$M(.,t)$ is a measurable function for all $t \geq 0$.

A function M which satisfies the conditions Φ_1 and Φ_2 is called a Musielak-Orlicz function. For a Musielak-Orlicz function M we put $M_x(t) = M(x,t)$ and we associate its non-negative reciprocal function M_x^{-1}, with respect to t, that is $M_x^{-1}(M(x,t)) = M(x,M_x^{-1}(t)) = t$.
Let M and P be two Musielak-Orlicz functions, we say that P grows essentially less rapidly than M at 0(resp. near infinity, and we write $P << M$, for every positive constant c, we have $\lim_{t\to 0}\Big(\sup_{x\in\Omega} \frac{P(x,ct)}{M(x,t)}\Big) = 0$ (resp.$\lim_{t\to\infty}\Big(\sup_{x\in\Omega} \frac{P(x,ct)}{M(x,t)}\Big) = 0\Big)$.

Remark 1 *[12] If $P << M$ near infinity, then $\forall\epsilon > 0$ there exist $k(\epsilon) > 0$ such that for almost all $x \in \Omega$ we have $P(x,t) \leq k(\epsilon)M(x,\epsilon t) \quad \forall t \geq 0$.*

2.2 Musielak-Orlicz space

For a Musielak-Orlicz function M and a mesurable function $u : \Omega \to I\!R$, we define the functionnal

$$\varrho_{M,\Omega}(u) = \int_\Omega M(x,|u(x)|)dx.$$

The set $K_M(\Omega) = \{u : \Omega \to I\!R \text{ mesurable} : \varrho_{M,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class. The Musielak-Orlicz space $L_M(\Omega)$ is the vector space generated by $K_M(\Omega)$; that is, $L_M(\Omega)$ is the smallest linear space containing the set $K_M(\Omega)$. Equivalently

$$L_M(\Omega) = \{u : \Omega \to I\!R \quad \text{mesurable} : \quad \varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

For any Musielak-Orlicz function M, we put $\overline{M}(x,s) = \sup_{t\geq 0}(st - M(x,s))$. $\overline{M}$ is called the Musielak-Orlicz function complementary to M (or conjugate of M) in the sense of Young with respect to s. We say that a sequence of function $u_n \in L_M(\Omega)$ is modular convergent to $u \in L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \varrho_{M,\Omega}(\frac{u_n-u}{\lambda}) = 0$.

This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$(see [17]).
In the space $L_M(\Omega)$, we define the following two norms

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M(x, \tfrac{|u(x)|}{\lambda})dx \leq 1\Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{M,\Omega} = \sup_{\|v\|_{\overline{M}}\leq 1} \int_\Omega |u(x)v(x)|dx,$$

where $\overline{M}$ is the Musielak-Orlicz function complementary to M. These two norms are equivalent [17]. $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is by denoted $E_M(\Omega)$. It is a separable space and $(E_M(\Omega))^* = L_M(\Omega)$. We have $E_M(\Omega) = K_M(\Omega)$, if and only if M satisfies the Δ_2−condition for large values of t or for all values of t, according to whether Ω has finite measure or not.
We define

$$W^1L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall\alpha \leq 1\},$$
$$W^1E_M(\Omega) = \{u \in E_M(\Omega) : D^\alpha u \in E_M(\Omega), \quad \forall\alpha \leq 1\},$$

where $\alpha = (\alpha_1, ..., \alpha_N)$,$|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_M(\Omega)$ is called the Musielak-Orlicz-Sobolev space.
Let $\overline{\varrho}_{M,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{M,\Omega}(D^\alpha u)$ and $\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{M,\Omega}(\frac{u}{\lambda}) \leq 1\}$ for $u \in W^1L_M(\Omega)$.
These functionals are convex modular and a norm on $W^1L_M(\Omega)$, respectively. Then pair $(W^1L_M(\Omega), \|u\|^1_{M,\Omega})$ is a Banach space if M satisfies the following condition (see [10]),

$$\text{There exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x\in\Omega} M(x,1) > c.$$

The space $W^1L_M(\Omega)$ is identified to a subspace of the product $\Pi_{\alpha\leq 1}L_M(\Omega) = \Pi L_M$. We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(I\!R)$ on Ω. The space $W^1_0L_M(\Omega)$ is defined as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$ and the space $W^1_0E_M(\Omega)$ as the(norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
For two complementary Musielak-Orlicz functions M and $\overline{M}$, we have [17].

- The Young inequality:

$$st \leq M(x,s) + \overline{M}(x,t) \text{ for all } s,t \geq 0 \text{ , } x \in \Omega.$$

- The Holder inequality

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \|u\|_{M,\Omega}|||v|||_{\overline{M},\Omega} \text{ for all } u \in L_M(\Omega), v \in L_{\overline{M}}(\Omega).$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ (respectively in $W^1_0L_M(\Omega)$) if, for some $\lambda > 0$.

$$\lim_{n\to\infty} \overline{\varrho}_{M,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

The following spaces of distributions will also be used

$$W^{-1}L_{\overline{M}}(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\leq 1}(-1)^\alpha D^\alpha f_\alpha$$

$$\text{where}\quad f_\alpha \in L_{\overline{M}}(\Omega)\Big\},$$

and

$$W^{-1}E_{\overline{M}}(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\leq 1}(-1)^\alpha D^\alpha f_\alpha$$

$$\text{where}\quad f_\alpha \in E_{\overline{M}}(\Omega)\Big\}.$$

Lemma 1 *[17] Let Ω be a bounded Lipschitz domain in $I\!R^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

- *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$,*
- *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have*

$$\frac{M(x,t)}{M(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \geq 1,$$

- *For all $y \in \Omega$, $\int_\Omega M(y,1)dx < \infty$,*
- *There exists a constant $C > 0$ such that*

$$\overline{M}(y,t) \leq C \quad a.e. \quad in \quad \Omega.$$

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence.
Consequently, the action of a distribution S in in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1L_M(\Omega)$ is well defined. It will be denoted by $< S, u >$.

2.3 Truncation Operator

T_k, $k > 0$, denotes the truncation function at level k defined on $I\!R$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2 *[12] Let $F : I\!R \to I\!R$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an Musielak-Orlicz function and let $u \in W_0^1L_M(\Omega)$(resp.$u \in W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$(resp.$u \in W_0^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(x)\frac{\partial u}{\partial x_i} & a.e.\ in\ \{x \in \Omega;\ u(x) \notin D\} \\ 0 & a.e.\ in\ \{x \in \Omega;\ u(x) \in D\} \end{cases}$$

Lemma 3 *Suppose that Ω satisfies the segement property and let $u \in W_0^1L_M(\Omega)$. Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that $u_n \to u$ for modular convergence in $W_0^1L_M(\Omega)$. Furthermore, if $u \in W_0^1L_M(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.*

Let Ω be an open subset of $I\!R^N$ and let M be a Musielak-Orlicz function satisfying

$$\int_0^1 \frac{M_x^{-1}(t)}{t^{\frac{N+1}{N}}}dt = \infty \quad \text{a.e.} \quad x \in \Omega, \tag{2}$$

and the conditions of Lemma (1). We may assume without loss of generality that

$$\int_0^1 \frac{M_x^{-1}(t)}{t^{\frac{N+1}{N}}}dt < \infty \quad \text{a.e.} \quad x \in \Omega. \tag{3}$$

Define a function $M^* : \Omega \times [0,\infty)$ by $M^*(x,s) = \int_0^s \frac{M_x^{-1}(t)}{t^{\frac{N+1}{N}}}dt$ $x \in \Omega$ and $s \in [0,\infty)$.
M^* its called the Sobolev conjugate function of M (see [18] for the case of Orlicz function).

Theorem 1 *Let Ω be a bounded Lipschitz domain and let M be a Musielak-Orlicz function satisfying (2),(3) and the conditions of lemma (1). Then $W_0^1L_M(\Omega) \hookrightarrow L_{M^*}(\Omega)$, where M^* is the Sobolev conjugate function of M. Moreover, if Φ is any Musielak-Orlicz function increasing essentially more slowly than M^* near infinity, then the imbedding $W_0^1L_M(\Omega) \hookrightarrow L_\phi(\Omega)$, is compact.*

Corollaire 1 *Under the same assumptions of theorem (1), we have $W_0^1L_M(\Omega) \hookrightarrow\hookrightarrow L_M(\Omega)$.*

Lemma 4 *If a sequence $u_n \in L_M(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 5 *Let $u_n, u \in L_M(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_M(\Omega), L_{\overline{M}}(\Omega))$.*

Démonstration: Let $\lambda > 0$ such that $\int_\Omega M(x, \frac{u_n-u}{\lambda})dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_{\overline{M}}(\Omega)$. Multiplying v by a suitable constant, we can assume $\lambda v \in \mathcal{L}_{\overline{M}}(\Omega)$. By Young's inequality,

$$|(u_n - u)v| \leq M(x, \frac{u_n - u}{\lambda}) + \overline{M}(x, \lambda v)$$

which implies, by Vitali's theorem, that $\int_\Omega |(u_n - u)v|dx \to 0$.

2.4 Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω an bounded open subset $I\!R^N$ and let $Q_T = \Omega \times]0, T[$ with some given $T > 0$. Let M be an Musielak-Orlicz function, for each $\alpha \in I\!N^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in I\!N^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T),$$
$$\forall \alpha \in I\!N^N, |\alpha| \leq 1\},$$
$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T),$$
$$\forall \alpha \in I\!N^N, |\alpha| \leq 1\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm $\|u\| = \sum_{|\alpha|\leq m} \|\nabla_x^\alpha u\|_{M,Q_T}$. We can easily show that they form a complementary system when Ω satisfies the Lipschitz domain [17]. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which have as many copies as there is α-order derivatives,$|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q_T)$ then the function : $t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then the concerned function is a $W^{1,x}E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds $W^{1,x}E_M(\Omega) \subset L^1(0,T,W^{1,x}E_M(\Omega))$. The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However,the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$, is in $L^1(0,T)$ .The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show as in [8],that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W_0^{1,x}E_M(Q_T)$, of some subsequence $(u_i) \subset \mathcal{D}(Q_T)$ for the modular convergence; i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$

$$\int_{Q_T} M(x, \frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda}) dx\,dt \to 0 \quad \text{as} i \to \infty. \tag{4}$$

This implies that (u_i) converge to u in $W^{1,x}L_M(Q_T)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
Consequently,

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}. \tag{5}$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$.
We have the following complementary system $\begin{pmatrix} W_0^{1,x}L_M(Q_T) & F \\ W_0^{1,x}E_M(Q_T) & F_0 \end{pmatrix}$ F being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is show that,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q_T}$ where the infimum is taken on all possible decompositions $f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q_T)$.

The space F_0 is then given by, $F_0 = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q_T) \right\}$ and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

Theorem 2 *[14] Let Ω be a bounded Lipchitz domain and let M be a Musielak-Orlicz function satisfying the same conditions of Theorem (1). Then there exists a constant $\lambda > 0$ such that $\|u\|_M \leq \lambda\|\nabla u\|_M$, $\forall \in W_0^1L_M(Q_T)$.*

Definition 1 *We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ for the modular convergence if we can write $u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0$ and $u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$ with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q_T)$ for modular convergence for all $|\alpha| \leq 1$ and $u_n^0 \to u^0$ strongly in $L^1(Q_T)$*

Lemma 6 *Let $\{u_n\}$ be a bounded sequence in $W^{1,x}L_M(Q_T)$ such that $\frac{\partial u_n}{\partial t} = \alpha_n + \beta_n$ in $\mathcal{D}'(Q_T)$, $u_n \rightharpoonup u$, weakly in $W^{1,x}L_M(Q_T)$, for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ with $\{\alpha_n\}$ and $\{\beta_n\}$ two bounded sequences respectively in $W^{-1,x}L_{\overline{M}}(Q_T)$ and in $\mathcal{M}(Q_T)$. Then $u_n \to u$ in $L^1_{loc}(Q_T)$. Furthermore, if $u_n \in W_0^{1,x}L_M(Q_T)$, then $u_n \to u$ strongly in $L^1(Q_T)$.*

Theorem 3 *if $u \in W^{1,x}L_M(Q_T) \cap L^1(Q_T)$ (resp. $W_0^{1,x}L_M(Q_T) \cap L^1(Q_T)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q}_T)$ (resp. $\mathcal{D}(\overline{I}, \mathcal{D}(Q_T))$) such that $v_j \to u$ in $W^{1,x}L_M(Q_T)$ and $\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ for the modular convergence.*

Démonstration: Let $u \in W^{1,x}L_M(Q_T) \cap L^1(Q_T)$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, then for any $\epsilon > 0$. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_{\overline{M}}(Q_T)$ for all $|\alpha| \leq 1$ and $u^0 \in L^1(Q_T)$, we will show that there exits $\lambda > 0$ (depending Only on u and N) and there exists $v \in \mathcal{D}(\overline{Q}_T)$ for which we can write $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\overline{Q}_T)$ such that

$$\int_{Q_T} M(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda}) dxdt \leq \epsilon, \forall |\alpha| \leq 1, \tag{6}$$

$$\|v - u\|_{L^1(Q_T)} \leq \epsilon, \tag{7}$$

$$\|v^0 - u^0\|_{L^1(Q_T)} \leq \epsilon, \tag{8}$$

$$\int_{Q_T} \overline{M}(x, \frac{v^\alpha - u^\alpha}{\lambda}) dxdt \leq \epsilon, \forall |\alpha| \leq 1 \tag{9}$$

The equation (6) flows from a slight adaptation of the arguments [17],The equations (7),(8) flows also from classical approximation results. For The equation (9) we know that $\mathcal{D}(\overline{Q}_T)$ is dense in $L_{\overline{M}}(Q_T)$ for the modular convergence. The case where $u \in W_0^{1,x}L_M(Q_T) \cap L^1(Q_T)$) can be handled similarly without essential difficulty as it mentioned [17].

Remark 2 *The assumption $u \in L^1(Q_T)$ in theorem (3) is needed only when Q_T has infinite measure, since else, we have $L_M(Q_T)) \subset L^1(Q_T)$ and so $W^{1,x}L_M(Q_T) \cap L^1(Q_T) = W^{1,x}L_M(Q_T)$.*

Remark 3 *If in the statement of theorem (3) above, one takes $I = I\!R$, we have that $\mathcal{D}(\Omega \times I\!R)$ is dense in $\{u \in W_0^{1,x}L_M(\Omega \times I\!R) \cap L^1\Omega \times I\!R) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times I\!R) + L^1(\Omega \times I\!R)\}$ for the modular convergence. This trivially follows from the fact that $\mathcal{D}(I\!R, \mathcal{D}(\Omega)) = \mathcal{D}(\Omega \times I\!R)$.*

Remark 4 *Let $a < b \in I\!R$ and Ω be a bounded open subset of $I\!R^N$ with the segment property, then* $\{u \in W_0^{1,x}L_M(\Omega \times (a,b)) \cap L^1\Omega \times (a,b)) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (a,b)) + L^1(\Omega \times (a,b))\} \subset \mathcal{C}([a,b], L^1(\Omega))$.

Démonstration: Let $u \in W_0^{1,x}L_M(\Omega \times (a,b))$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (a,b)) + L^1(\Omega \times (a,b))$.
After two consecutive reflections first with respect to $t = b$ and then with respect to $t = a$, $\hat{u}(x,t) = u(x,t)\chi_{(a,b)} + u(x, 2b-t)\chi_{(b,2b-a)}$ in $\Omega \times (b, 2b-a)$ and $\tilde{u}(x,t) = \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x, 2a-t)\chi_{(3a-2b,a)}$ in $\Omega \times (3a-2b, 2b-a)$. We get function $\tilde{u} \in W_0^{1,x}L_M(\Omega \times (3a-2b, 2b-a))$ with $\frac{\partial \tilde{u}}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times (3a-2b, 2b-a)) + L^1(\Omega \times (3a-2b, 2b-a))$. Now by letting a function $\eta \in \mathcal{D}(I\!R)$ with $\eta = 1$ on $[a,b]$ and $supp\eta \subset (3a-2b, 2b-a)$, we set $\overline{u} = \eta\tilde{u}$, therefore,by standard arguments (see [19]), we have $\overline{u} = u$ on $(\Omega \times (a,b))$, $\overline{u} \in W_0^{1,x}L_M(\Omega \times I\!R) \cap L^1(\Omega \times I\!R)$ and $\frac{\partial \overline{u}}{\partial t} \in W_0^{-1,x}L_{\overline{M}}(\Omega \times I\!R) + L^1(\Omega \times I\!R)$.
Let now v_j the sequence given by theorem (3) corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \quad \text{in} \quad W_0^{1,x}L_M(\Omega \times I\!R)$$

and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \quad \text{in} \quad W_0^{-1,x}L_{\overline{M}}(\Omega \times I\!R) + L^1(\Omega \times I\!R)$$

for the modular convergence.
If we denote $S_k(s) = \int_0^s T_k(t)dt$ the primitive of T_k. We have, $\int_\Omega S_1(v_i - v_j)(\tau)dx = \int_\Omega \int_{-\infty}^r T_1(v_i - v_j)(\frac{\partial v_i}{\partial t} - \frac{\partial v_j}{\partial t})dxdt \to 0$ as $i, j \to 0$, from which, one deduces that v_j is a Cauchy sequence in $C(I\!R; L^1(\Omega))$ and hence $\overline{u} \in C(I\!R, L^1(\Omega))$. Consequently, $u \in C([a;b]; L1(\Omega))$.

3 Formulation of the problem and main results

Let Ω be an open subset of $I\!R^N$ $(N \geq 2)$ satisfying the segment property,and let M and P be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfies conditions of Lemma 1, M is decreasing in x and $P \ll M$.
$b : \Omega \times I\!R \to I\!R$ is a Carathéodory function such that for every, $x \in \Omega$, $b(x,.)$ is a strictly increasing $\mathcal{C}^1(I\!R)$-function and

$$b \in L^\infty(\Omega \times I\!R) \quad \text{with} \quad b(x,0) = 0, \tag{10}$$

There exists a constant $\lambda > 0$ and functions $A \in L^\infty(\Omega)$ and $B \in L_M(\Omega)$ such that

$$\lambda \leq \frac{\partial b(x,s)}{\partial s} \leq A(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B(x) \tag{11}$$

a.e. $x \in \Omega$ and $\forall\ |s| \in I\!R$.
$A : D(A) \subset W_0^1L_M(Q_T) \to W^{-1}L_{\overline{M}}(Q_T)$ defined by $A(u) = -\text{div}\ a(x,t,u,\nabla u)$, where $a : Q_T \times I\!R \times I\!R^N \to I\!R^N$ is caratheodory function such that for a.e. $x \in \Omega$ and for all $s \in I\!R, \xi, \xi^* \in I\!R^N, \xi \neq \xi^*$

$$|a(x,t,s,\xi)| \leq \nu(a_0(x,t) + \overline{M}_x^{-1}P(x,|s|)), \tag{12}$$

with $a_0(.,.) \in E_{\overline{M}}(Q_T)$,

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0, \tag{13}$$

$$a(x,t,s,\xi).\xi \geq \alpha M(x,|\xi|) + M(x,|s|). \tag{14}$$

$\Phi(x,s,\xi) : Q_T \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \leq c(x,t)\overline{M}_x^{-1}M(x,\alpha_0|s|), \tag{15}$$

where $c(.,.) \in L^\infty(Q_T)$, $\|c(.,.)\|_{L^\infty(Q_T)} \leq \alpha$, and $0 < \alpha_0 < \min(1, \frac{1}{\alpha})$.
$H(x,t,s,\xi) : Q_T \times I\!R \times I\!R^N \to I\!R$ is a Carathéodory function such that

$$|H(x,t,s,\xi)| \leq h(x,t) + \rho(s)M(x,|\xi|), \tag{16}$$

$\rho : I\!R \to I\!R^+$ is continuous positive function which belong $L^1(I\!R)$ and $h(.,.)$ belong $L^1(Q_T)$.

$$f \in L^1(\Omega), \tag{17}$$

and

$$u_0 \in L^1(\Omega) \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{18}$$

Note that <,> means for either the pairing between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T))$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$.
Weak entropy solution: The definition of a entropy solution of Problem (1) can be stated as follows,

Definition 2 *A measurable function u defined on Q_T is a entropy solution of Problem* (1), *if it satisfies the following conditions:*

$$b(x,u) \in L^\infty(0,T; L^1(\Omega)), b(x,u)(t=0) = b(x,u_0) \quad in \quad \Omega,$$

$$T_k(u) \in W_0^{1,x}L_M(Q_T), \quad \forall k > 0, \quad \forall t \in]0,T],$$

$$\begin{cases} \int_0^T \left\langle \frac{\partial v}{\partial s}; \int_0^u \frac{\partial b(x,z)}{\partial s} T_k'(z-v)dz \right\rangle ds \\ + \int_\Omega \int_0^{u_0} \frac{\partial b(x,s)}{\partial s} T_k(s - v(0))dsdx \\ + \int_{Q_T} a(u,\nabla u)\nabla T_k(u-v)dxds + \int_{Q_T} \Phi(u)\nabla T_k(u-v)dxds \\ + \int_{Q_T} H(u,\nabla u)T_k(u-v)dxds \leq \int_{Q_T} f T_k(u-v)dxds \\ \forall k > 0, \quad and \quad \forall v \in W^{1,x}L_M(Q_T) \cap L^\infty(Q_T) \quad with \\ \quad v(T) = 0, \quad such\ that \quad \frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T) \end{cases} \tag{19}$$

Theorem 4 *Assume that* (11) − (18) *hold true . Then there exists at least one solution u of the following problem* (19).

4 Proof of theorem 4

Truncated problem.
For each $n > 0$, we define the following approximations

$$b_n(x,s) = b(x,T_n(s)) + \frac{1}{n}\, s \quad \forall\, r \in I\!R, \tag{20}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T, \tag{21}$$

$\forall\, s \in I\!R,\ \forall\, \xi \in I\!R^N$,

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\, s \in I\!R, \tag{22}$$

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}, \tag{23}$$

$$f_n \in L^1(Q_T) \text{ such that } f_n \to f \text{ strongly in } L^1(Q_T), \tag{24}$$

and $\|f_n\|_{L^1(Q_T)} \le \|f\|_{L^1(Q_T)}$,
and

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n(x,u_{0n}) \to b(x,u_0) \tag{25}$$

strongly in $L^1(\Omega)$.
Let us now consider the approximate problem :

$$\begin{cases} \dfrac{\partial b_n(x,u_n)}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) - div(\Phi_n(x,t,u_n)) \\ +H_n(x,u_n,\nabla u_n) = f_n \ in\ Q_T, \\ u_n(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_n(x,u_n)(t=0) = b_n(x,u_{0n}) \quad in \quad \Omega. \end{cases} \tag{26}$$

Since H_n is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^{1,x}L_M(Q_T)$ of (26)(see [20]).

Remark 5 *the explicit dependence in x and t of the functions a, Φ and H will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$, $\Phi(x,t,u) = \Phi(u)$ and $H(x,t,u,\nabla u) = H(u,\nabla u)$.*

Proposition 1 *let u_n be a solution of approximate equation* (26)*such that*

$$\begin{cases} T_k(u_n) \to T_k(u) \quad weakly\ in \quad W^{1,x}L_M(Q_T), \\ u_n \to u \quad a.e.\ in \quad Q_T, \\ b_n(x,u_n) \to b(x,u) \quad a.e.\ in \quad Q_T \quad and \quad b(x,u) \in L^\infty(Q_T), \\ a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \rightharpoonup a(T_k(u),\nabla T_k(u))\nabla T_k(u) \\ \quad weakly\ in \quad L^1(Q_T), \\ \nabla u_n \to \nabla u \quad a.e.\ in \quad Q_T, \\ H_n(u_n,\nabla u_n) \to H(u,\nabla u) \quad strongly\ in \quad L^1(Q_T). \end{cases} \tag{27}$$

then u be a solution of problem (19).

Démonstration: Let $v \in W_0^1L_M(Q_T) \cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ with $v(T) = 0$, then by theorem 3 we can take $\overline{v} = v$ on Q_T, $\overline{v} \in W^{1,x}L_M(\Omega \times I\!R) \cap L^1(\Omega \times I\!R) \cap L^\infty(\Omega \times I\!R)$, $\frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, and there exists $v_j \in \mathcal{D}(\Omega \times I\!R)$ such that $v_j(T) = 0$, $v_j \to \overline{v}$ in $W_0^{1,x}L_M(\Omega \times I\!R)$ and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T), \tag{28}$$

for the modular convergence in $W_0^1L_M(Q_T)$, with $\|v_j\|_{L^\infty(Q_T)} \le (N+2)\|v\|_{L^\infty(Q_T)}$.

Pointwise multiplication of the approximate equation (26) by $T_k(u_n - v_j)$, we get

$$\begin{cases} \displaystyle\int_0^T < \frac{\partial b_n(x,u_n)}{\partial s}; T_k(u_n - v_j) > ds \\ +\displaystyle\int_{Q_T} a_n(u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds \\ +\displaystyle\int_{Q_T} \Phi_n(u_n)\nabla T_k(u_n - v_j)dxds \\ +\displaystyle\int_{Q_T} H_n(u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds \\ = \displaystyle\int_{Q_T} f_n T_k(u_n - v_j)dxds \end{cases} \tag{29}$$

We pass to the limit as in (29), n tend to $+\infty$ and j tend to $+\infty$:
Limit of the first term of (29):
The first term can be written

$$\int_0^T < \frac{\partial b_n(x,u_n)}{\partial s}; T_k(u_n - v_j) > ds$$

$$= \int_0^T < \frac{\partial v}{\partial s}; \int_0^{u_n} \frac{\partial b_n(x,z)}{\partial s} T_k'(z - v_j) > ds$$

$$+ \int_\Omega \int_0^{u_n(T)} \frac{\partial b_n(x,s)}{\partial s} T_k(s - v_j(T))dsdx$$

$$- \int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,s)}{\partial s} T_k(s - v_j(0))dsdx,$$

the fact that $\frac{\partial b_n(x,s)}{\partial s} \ge 0$ and $v_j(T) = 0$, we get

$$\int_\Omega \int_0^{u(T)} \frac{\partial b(x,s)}{\partial s} T_k(s - v_j(T))dsdx =$$

$$\int_\Omega \int_0^{u(T)} \frac{\partial b(x,s)}{\partial s} T_k(s)dsdx \ge 0$$

On the other hand, we have u_{0n} converge to u_0 strongly in $L^1(\Omega)$, then
$\lim_{n\to+\infty} \int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,s)}{\partial s} T_k(s - v_j(0))dsdx$

$$= \int_\Omega \int_0^{u_0} \frac{\partial b(x,s)}{\partial s} T_k(s - v_j(0))dsdx,$$

with $M = k + (N+2)\|v\|_\infty$ and $T_M(u_n)$ converges to $T_M(u)$ strongly in $E_M(Q_T)$, we obtain

$$\lim_{n\to+\infty} \int_0^T < \frac{\partial v_j}{\partial t}; \int_0^{u_n} \frac{\partial b_n(x,z)}{\partial s} T_k'(z - v_j)dz > ds$$

$$= \lim_{n\to+\infty} \int_0^T < \frac{\partial v_j}{\partial s}; \int_0^{T_M(u_n)} \frac{\partial b_n(x,z)}{\partial s} T_k'(z - v_j)dz > ds$$

$$= \int_0^T < \frac{\partial v_j}{\partial s}; \int_0^{T_M(u)} \frac{\partial b(x,z)}{\partial s} T_k'(z - v_j)dz > ds,$$

then

$$\int_0^T < \frac{\partial v_j}{\partial s}; \int_0^{T_M(u)} \frac{\partial b(x,z)}{\partial s} T_k'(z - v_j)dz > ds$$

$$\le \lim_{n\to+\infty} \int_0^T < \frac{\partial b(x,u_n)}{\partial s} T_k(u_n - v_j) > ds$$

$$+ \int_\Omega \int_0^{u_0} < \frac{\partial b(x,s)}{\partial s} T_k(s - v_j(0)) > dsdx,$$

using (28), the definition of T_k and pass to limit as $j \to +\infty$, we deduce

$$\int_0^T <\frac{\partial v}{\partial s};\int_0^{T_M(u)}\frac{\partial b(x,z)}{\partial s}T_k'(z-v)dz> ds$$
$$\leq \lim_{j\to+\infty}\lim_{n\to+\infty}\int_0^T <\frac{\partial b(x,u_n)}{\partial s}T_k(u_n-v_j)> ds$$
$$+\int_\Omega\int_0^{u_0} <\frac{\partial b(x,t)}{\partial s}T_k(s-v(0))> dsdx.$$

- We can follow same way in [21]to prove that

$$\liminf_{j\to\infty}\liminf_{n\to\infty}\int_{Q_T} a(u_n,\nabla u_n)\nabla T_k(u_n-v_j)dxds$$
$$\geq \int_{Q_T} a(u,\nabla u)\nabla T_k(u-v)dxds.$$

- For $n \geq k+(N+2)\|v\|_{L^\infty(Q_T)}$ $\Phi_n(u_n)\nabla T_k(u_n-v_j) = \Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n-v_j)$. The pointwise convergence of u_n to u as n tends to $+\infty$ and (15), then $\Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n-v_j) \rightharpoonup \Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v_j)$ weakly for $\sigma(\Pi L_M,\Pi L_{\overline{M}})$.
  In a similar way, we obtain

$$\lim_{j\to\infty}\int_{Q_T}\Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v_j)dxds$$
$$=\int_{Q_T}\Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v)dxds$$
$$=\int_{Q_T}\Phi(u)\nabla T_k(u-v)dxds.$$

- Limit of $H_n(u_n,\nabla u_n)T_k(u_n-v_j)$:
  Since $H_n(u_n,\nabla u_n)$ converge strongly to $H(x,s,u,\nabla u)$ in $L^1(Q_T)$ and the pointwise convergence of u_n to u as $n\to+\infty$, it is possible to prove that $H_n(u_n,\nabla u_n)T_k(u_n-v_j)$ converge to $H(u,\nabla u)T_k(u-v_j)$ in $L^1(Q_T)$ and $\lim_{j\to\infty}\int_{Q_T}H(u,\nabla u)T_k(u-v_j)dxds = \int_{Q_T}H(u,\nabla u)T_k(u-v)dxds.$

- Since f_n converge strongly to f in $L^1(Q_T)$, and $T_k(u_n-v_j)\to T_k(u_n-v_j)$ weakly* in $L^\infty(Q_T)$, we have $\int_{Q_T} f_nT_k(u_n-v_j)dxds \to \int_{Q_T} fT_k(u-v_j)dxds$ as $n\to\infty$ and also we have $\int_{Q_T} fT_k(u-v_j)dxds \to \int_{Q_T} fT_k(u-v)dxds$ as $j\to\infty$
  Finally, the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (29) and to conclude that u satisfies (19).

It remains to show that $b(x,u)$ satisfies the initial condition. In fact, remark that, $B_M(x,u_n) = \int_0^{u_n}\frac{\partial b(x,s)}{\partial s}T_M(s-v)ds$ is bounded in $L^\infty(Q_T)$. Secondly, by (69) we show that $\frac{\partial B_M(x,u_n)}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T))+L^1(Q_T)$. As a consequence, a Lemma 4 implies that $B_M(x,u_n)$ lies in a compact set of $C^0([0,T];L^1(\Omega))$. It follows that, $B_M(x,u_n)(t=0)$ converges to $B_M(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the order hand, the smoothness of B_M imply that $B_M(x,u_n)(t=0)$ converges to $B_M(x,u)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $B_M(x,u_n)(t=0)=B_M(x,u_{0n})$ converges to $B_M(x,u)(t=0)$ strongly in $L^1(\Omega)$, we obtain $B_M(x,u)(t=0)=B_M(x,u_0)$ a.e. in Ω and for all $M>0$, now letting M to $+\infty$, we conclude that $b(x,u)(t=0)=b(x,u_0)$ a.e. in Ω.

Remark 6 *We focus our work to show the conditions of the proposition 27, then for this we go through 4 steps to arrive at our result.*

Step 1: In this step let us begin by showing

Lemma 7
Let $\{u_n\}_n$ be a solution of the approximate problem (26), then for all $k>0$, there exists a constants C_1 and C_2 such that

$$\int_{Q_T} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq kC_1, \quad (30)$$

and

$$\int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \leq kC_2, \quad (31)$$

where C_1 and C_2 does not depend on the n and k.

Démonstration: Fixed $k>0$,
Let $\tau\in(0,T)$ and using $\exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}$ as a test function in problem (26), where $G(s)=\int_0^s\frac{\rho(r)}{\alpha'}dr$ and $\alpha'>0$ is a parameter to be specified later, we get:

$$\int_{Q_\tau}\frac{\partial b_n(x,u_n)}{\partial s}\exp(G(u_n))T_k(u_n)^+\chi_{(0,t)}dxdt \quad (32)$$
$$+\int_{Q_\tau} a(u_n,\nabla u_n)\frac{\rho(u_n)}{\alpha'}\exp(G(u_n))\nabla u_nT_k(u_n)^+dxdt \quad (33)$$
$$+\int_{\{0\leq u_n\leq k\}} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)dxdt \quad (34)$$
$$+\int_{Q_\tau}\Phi_n(u_n)\nabla\Big(\exp(G(u_n))T_k(u_n)^+\Big)dxdt \quad (35)$$
$$+\int_{Q_\tau} H(u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt \quad (36)$$
$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|f_n\|_{L^1(Q_T)}. \quad (37)$$

For the (32), we have

$$\int_{Q_\tau}\frac{\partial b_n(x,u_n)}{\partial s}\exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}dxdt$$
$$=\int_\Omega B_{n,k}(x,u_n(\tau))dx-\int_\Omega B_{n,k}(x,u_n(0))dx,$$

where

$$B_{n,k}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(G(T_k(s)))T_k(s)^+ ds.$$

By (11), we have $\int_\Omega B_{n,k}(x,u_n(\tau))dx \geq 0$ and

$$\int_{Q_\tau} B_{n,k}(x,u_n(0))dx \leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|b(x,u_0\|_{L^1(\Omega)}.$$

For the (35), if we use (15) and Young inequality , we get

$$\int_{Q_\tau} \Phi_n(u_n))\nabla(\exp(G(u_n))T_k(u_n)^+)dxdt \leq$$

$$\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big[\alpha_0\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$+\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\Big]$$

$$+\|c(.,.)\|_{L^\infty(Q_T)}\alpha_0\int_{Q_\tau} M(x,u_n)\exp(G(u_n))dxdt$$

$$+\|c(.,.)\|_{L^\infty(Q_T)}\int_{Q_\tau} M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dxdt.$$

For the (36), we have,

$$\int_{Q_\tau} H_n(u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_{Q_T}|h(x,t)|dxdt$$

$$+\int_{Q_\tau}\rho(u_n)\exp(G(u_n))M(x,\nabla T_k(u_n))T_k(u_n)^+dxdt.$$

finally using the previous inequalities and (14), we obtain

$$\begin{cases} \frac{1}{\alpha'}\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\frac{\alpha}{\alpha'}\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ \leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}[\alpha_0\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt] \\ +\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}\int_{\{0\leq u_n\leq k\}} M(x,u_n)\exp(G(u_n))dxdt \\ +\|c(.,.)\|_{L^\infty(Q_T)}\int_{Q_\tau} M(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt \\ +\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +k[\exp(\frac{\|\rho\|_{L^1}}{\alpha'}(\|f\|_{L^1(Q_T)}+\|b(x,u_0\|_{L^1(\Omega)} \\ +\int_{Q_T}|h(x,t)|dxdt], \end{cases} \quad (38)$$

which becomes after simplification,

$$\begin{cases} [\frac{1-\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}]\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +[\frac{\alpha-\|c(.,.)\|_{L^\infty(Q_T)}-\alpha'}{\alpha'}]\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ \leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}[\alpha_0\alpha\int_{\{0\leq u_n\leq k\}} M(x,u_n)\exp(G(u_n))dxdt \\ +\alpha M(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt]+kC. \end{cases} \quad (39)$$

If we choose α' such that $\alpha' < \alpha - \|c(.,.)\|_{L^\infty(Q_T)}$ and using again (14) in (39) we get

$$[1-\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}]\int_{Q_\tau} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \leq kC. \quad (40)$$

we deduce,

$$\int_{\{0\leq u_n\leq k\}} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)dxdt \leq kc_1.$$

one has $\exp(G(u_n)) \geq 1$ for in $\{(x,t)\in Q_T : 0\leq u_n\leq k\}$ then

$$\int_{\{0\leq u_n\leq k\}} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1. \quad (41)$$

and by (14) another again

$$\int_{Q_t} M(x,|\nabla T_k(u_n)^+|)dxdt \leq kc_2. \quad (42)$$

Similarly, taking $\exp(-G(u_n)T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in problem (26), we get

$$\int_{Q_\tau}\frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))T_k(u_n)^-dxdt \quad (43)$$

$$+\int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \quad (44)$$

$$+\int_{Q_\tau}\Phi_n(u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \quad (45)$$

$$+\int_{Q_\tau} H(u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n)^-dxdt \quad (46)$$

$$\geq \int_{Q_\tau} f_n\exp(-G(u_n))T_k(u_n)^-dxdt. \quad (47)$$

and using same techniques above, we obtain

$$\int_{\{-k\leq u_n\leq 0\}} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1. \quad (48)$$

since $\exp(-G(u_n)) \geq 1$ in $\{(x,t)\in Q_T : -k\leq u_n\leq 0\}$ and

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n)^-|)dxdt \leq kc_2. \quad (49)$$

Combining now (41) and (48) we get,

$$\int_{Q_\tau} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kC_1, \quad (50)$$

in the same with (42) and (49) we get,

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n)|)dxdt \leq kC_2. \quad (51)$$

we conclude that $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ independently of n, and for any $k>0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^{1,x}L_M(Q_T). \tag{52}$$

On the other hand, using (51), we have

$$M(x,\frac{k}{\delta})meas\{|u_n|>k\} \leq \int_{\{|u_n|>k\}} M(x,\frac{|T_k(u_n)|}{\delta})dxdt$$

$$\leq \int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \leq kC_2,$$

then

$$meas\{|u_n|>k\} \leq \frac{kC_2}{M(x,\frac{k}{\delta})},$$

for all n and for all k.
Assuming that there exists a positive function M such that $\lim_{t\to\infty}\frac{M(t)}{t}=+\infty$ and $M(t)\leq ess\inf_{x\in\Omega} M(x,t)$, $\forall t\geq 0$. thus, we get

$$\lim_{k\to\infty} meas\{|u_n|>k\}=0.$$

Now we turn to prove the almost every convergence of u_n, $b_n(x,u_n)$ and $a_n(x,t,T_k(u_n),\nabla T_k(u_n))$.

Proposition 2 *Let u_n be a solution of the approximate problem, then*

$$u_n \to u \quad a.e\ in \quad Q_T, \tag{53}$$

$b_n(x,u_n)\to b(x,u)$ *a.e in* Q_T *and*

$$b(x,u)\in L^\infty(0,T,L^1(\Omega)), \tag{54}$$

$$a_n(T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k \quad in \quad (L_{\overline{M}}(Q_T))^N, \quad for \quad \sigma(\Pi L_{\overline{M}},\Pi E_M), \tag{55}$$

for some $\varpi_k\in(L_{\overline{M}}(Q_T))^N$.

Démonstration: :
Proof of (53) **and** (54)**:**
Proceeding as in [22], we have for any $S\in W^{2,\infty}(I\!R)$, such that S', has a compact support ($suppS'\subset[-K,K]$).

$$B_S^n(x,u_n) \quad \text{is bounded in} \quad W_0^{1,x}L_M(Q_T), \tag{56}$$

and

$$\frac{\partial B_S^n(x,u_n)}{\partial t} \quad \text{is bounded in} \quad L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T), \tag{57}$$

independently of n.
Indeed, we have first

$$|\nabla B_S^n(x,u_n)| \leq \|A_K\|_{L^\infty(\Omega)}|DT_k(u_n)|\|S'\|_{L^\infty(\Omega)}+K\|S'\|_{L^\infty(\Omega)}B_K(x) \tag{58}$$

a.e. in Q_T.
As a consequence of (58) and (51) we then obtain (56). To show that (57) holds true, we multiply the equation (26) by $S'(u_n)$, to obtain

$$\frac{\partial B_S^n(x,u_n)}{\partial t} = div(S'(u_n)a_n(u_n,\nabla u_n)) - S''(u_n)a_n(u_n,\nabla u_n)\nabla u_n$$
$$+div(S'(u_n)\Phi_n(u_n)) - S''(u_n)\Phi_n(u_n)\nabla u_n$$
$$+H_n(u_n,\nabla u_n)S'(u_n)+f_nS'(u_n) \quad in \quad \mathcal{D}(Q_T). \tag{59}$$

where $B_S^n(x,r)=\int_0^r S'(s)\frac{\partial b_n(x,s)}{\partial s}ds$. Since $suppS'$ and $suppS''$ are both included in $[-K,K]$, u_n may be replaced by $T_k(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (59) is bounded either in $W^{-1,x}L_{\overline{M}}(Q_T)$ or in $L^1(Q_T)$ which shows that (57) holds true.
Arguing again as in [22] estimates (56), (57) and the following remark (1), we can show (53) and (54).
Proof of (55) **:**
The same way in [15], we deduce that $a_n(T_k(u_n),\nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$, and we obtain (55).
Step 2: This technical lemma will help us in the step 3 of the demonstration,

Lemma 8 *If the subsequence u_n satisfies* (26), *then*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{60}$$

Démonstration: Taking the function $Z_m(u_n)=T_1(u_n-T_m(u_n))^-$ and multiplying the approximating equation (26) by the test function $\exp(-G(u_n))Z_m(u_n)$ we get

$$\begin{cases} \int_{Q_T} B_{n,m}(x,u_n(T))dx \\ +\int_{Q_T} a_n(u_n,\nabla u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx\,dt \\ +\int_{Q_T}\Phi_n(u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx\,dt \\ +\int_{Q_T} H_n(u_n,\nabla u_n)\exp(-G(u_n))Z_m(u_n)dx\,dt \\ =\int_{Q_T} f_n\exp(-G(u_n))Z_m(u_n)\,dx\,dt+\int_{Q_T} B_{n,m}(x,u_{0n})dx \end{cases} \tag{61}$$

where $B_{n,m}(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(-G(s))Z_m(s)ds.$
Using the same argument in step 2, we obtain

$$\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt \leq C(\int_{Q_T}|h(x,t)|Z_m(u_n)dxdt$$
$$+\int_{Q_T} f_nZ_m(u_n)dxdt+\int_{|u_{0n}|>m}|b_n(x,u_{0n})|dx).$$

where

$$C=\exp(\frac{\|\rho\|_{L^1}}{\alpha'})(\frac{\alpha}{\alpha-\|c(.,.)\|_{L^\infty(Q_T)}}).$$

Passing to limit as $n\to+\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n and $b_n(x,u_{0n})$ we get

$$\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt \leq C(\int_{Q_T} fZ_m(u)dxdt$$
$$+\int_{Q_T}|h(x,t)|Z_m(u)dxdt+\int_{\{|u_0|>m\}}|b(x,u_0)|dx).$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt = 0, \quad (62)$$

On the other hand, by (15) and Young inequality,for $n > m+1$ we obtain
$\int_{Q_T} |\Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)|dx\,dt$

$$\le \exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\int_{\{-(m+1)\le u_n\le -m\}} M(x,\alpha_0|T_{m+1}(u_n)|)dxdt$$

$$+\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt].$$

Using the pointwise convergence of u_n and by Lebesgues theorem, it follows,
$\lim_{n\to+\infty}\int_{Q_T} |\Phi_n(u_n)\exp(-G(u_n))\nabla Z_m(u_n)|dx\,dt$

$$\le \exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\int_{\{-(m+1)\le u\le -m\}} M(x,\alpha_0|T_{m+1}(u)|)dxdt$$

$$+\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt]$$

passing to the limit in as $m \to +\infty$, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \Phi_n(u_n)\exp(-G(u_n))\nabla Z_m(u_n)dx\,dt = 0.$$

Finally passing to the limit in (61), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{-(m+1)\le u_n\le -m\}} a_n(u_n,\nabla u_n)\nabla u_n dx\,dt = 0,$$

In the same way we take $Z_m(u_n) = T_1(u_n - T_m(u_n))^+$ and multiplying the approximating equation (26) by the test function $\exp(G(u_n))Z_m(u_n)$ and we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le u_n\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dx\,dt = 0,$$

on the above we get (60).

Step 3: Almost everywhere convergence of the gradients.

This step is devoted to introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method.

Lemma 9 *(See [23]) Under assumptions* (11)-(18), *and let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that:*

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M(Q_T),\Pi E_{\overline{M}}(Q_T)), \quad (63)$$

$$(a(x,t,z_n,\nabla z_n)) \quad is\ bounded\ in\ (L_{\overline{M}}(Q_T))^N, \quad (64)$$

$$\int_{Q_T} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dx\,dt \to 0, \quad (65)$$

as n and s tend to $+\infty$, *and where* χ_s *is the characteristic function of* $Q_s = \{(x,t) \in Q_T; |\nabla z| \le s\}$ *then,*

$$\nabla z_n \to \nabla z \quad a.e.\ in\ Q_T, \quad (66)$$

$$\lim_{n\to+\infty}\int_{Q_T} a(x,t,z_n,\nabla z_n)\nabla z_n dx\,dt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dx\,dt, \quad (67)$$

$$M(x,|\nabla z_n|) \to M(x,|\nabla z|) \quad in\ L^1(Q_T). \quad (68)$$

Let $v_j \in \mathcal{D}(Q_T)$ be a sequence such that $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence.
This specific time regularization of $T_k(v_j)$ (for fixed $k \ge 0$) is defined as follows.
Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_0^\mu \in L^\infty(\Omega)\cap W_0^1 L_M(\Omega) \quad \text{for all} \quad \mu > 0, \quad (69)$$

$$\|\alpha_0^\mu\|_{L^\infty(\Omega)} \le k, \text{ for all } \quad \mu > 0,$$

and α_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|\alpha_0^\mu\|_{M,\Omega}$ converges to 0 as $\mu \to +\infty$.
For $k \ge 0$ and $\mu > 0$, let us consider the unique solution $(T_k(v_j))_\mu \in L^\infty(Q_T)\cap W_0^{1,x}L_M(Q_T)$ of the monotone problem:

$$\frac{\partial(T_k(v_j))_\mu}{\partial t} + \mu((T_k(v_j))_\mu - T_k(v_j)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(v_j))_\mu(t=0) = \alpha_0^\mu \text{ in } \Omega.$$

Remark that due to

$$\frac{\partial(T_k(v_j))_\mu}{\partial t} \in W_0^{1,x}L_M(Q_T).$$

We just recall that, $(T_k(v_j))_\mu \to T_k(u)$ a.e. in Q_T, weakly-$*$ in $L^\infty(Q_T)$,

$(T_k(v_j))_\mu \to (T_k(u))_\mu$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence as $j \to +\infty$ and
$(T_k(u))_\mu \to T_k(u)$ in $W_0^{1,x}L_M(Q_T)$, for the modular convergence as $\mu \to +\infty$.

$$\|(T_k(v_j))_\mu\|_{L^\infty(Q_T)} \le max(\|(T_k(u))\|_{L^\infty(Q_T)}, \|\alpha_0^\mu\|_{L^\infty(\Omega)}) \le k,$$

$\forall\ \mu > 0$,$\forall\ k > 0$.
We introduce a sequence of increasing $\mathbf{C}^1(I\!R)$-functions S_m such that $S_m(r) = 1$ for $|r| \le m$, $S_m(r) = m+1-|r|$, for $m \le |r| \le m+1$, $S_m(r) = 0$ for $|r| \ge m+1$ for any $m \ge 1$ and we denote by $\epsilon(n,\mu,\eta,j,m)$ the quantities such that

$$\lim_{m\to+\infty}\lim_{j\to+\infty}\lim_{\eta\to+\infty}\lim_{\mu\to+\infty}\lim_{n\to+\infty}\epsilon(n,\mu,\eta,j,m) = 0,$$

the main estimate is

For fixed $k \ge 0$, let $W_{\mu,\eta}^{n,j} = T_\eta(T_k(u_n) - T_k(v_j)_\mu)^+$ and $W_{\mu,\eta}^{j} = T_\eta(T_k(u) - T_k(v_j)_\mu)^+$.
Multiplying the approximating equation by $\exp(G(u_n)))W_{\mu,\eta}^{n,j}S_m(u_n)$ and using the same technique in step 2 we obtain:

$$\begin{cases} \int_{Q_T} < \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt \\ + \int_{Q_T} a_n(u_n,\nabla u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta}) S_m(u_n)\,dx\,dt \\ + \int_{Q_T} a_n(u_n,\nabla u_n)\nabla u_n \exp(G(u_n)) W^{n,j}_{\mu,\eta} S'_m(u_n)\,dx\,dt \\ - \int_{Q_T} \Phi_n(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta}) S_m(u_n)\,dx\,dt \\ - \int_{Q_T} \Phi_n(u_n)\nabla u_n \exp(G(u_n)) W^{n,j}_{\mu,\eta} S'_m(u_n)\,dx\,dt \\ \le \int_{Q_T} f_n \exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt \\ + \int_{Q_T} h(x,t)\exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt. \end{cases} \tag{70}$$

Now we pass to the limit in (70) for k real number fixed.

In order to perform this task we prove below the following results for any fixed $k \ge 0$:

$$\int_{Q_T} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt \ge \epsilon(n,\mu,\eta,j) \tag{71}$$

for any $m \ge 1$,

$$\int_{Q_T} \Phi_n(u_n) S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt = \epsilon(n,j,\mu) \tag{72}$$

for any $m \ge 1$,

$$\int_{Q_T} \Phi_n(u_n)\nabla u_n S'_m(u_n)\exp(G(u_n)) W^{n,j}_{\mu,\eta}\,dx\,dt = \epsilon(n,j,\mu) \tag{73}$$

for any $m \ge 1$,

$$\int_{Q_T} a_n(u_n,\nabla u_n)\nabla u_n S'_m(u_n)\exp(G(u_n)) W^{n,j}_{\mu,\eta}\,dx\,dt \le \epsilon(n,m), \tag{74}$$

$$\int_{Q_T} a_n(u_n,\nabla u_n) S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt \le C\eta + \epsilon(n,j,\mu,m), \tag{75}$$

$$\int_{Q_T} f_n S_m(u_n)\exp(G(u_n)) W^{n,j}_{\mu,\eta}\,dx\,dt + \int_{Q_T} h(x,t)\exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt \le C\eta + \epsilon(n,\eta), \tag{76}$$

$$\int_{Q_T} [a(T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \times [\nabla T_k(u_n) - \nabla T_k(u)]\,dx\,dt \to 0. \tag{77}$$

Proof of (71):

Lemma 10

$$\int_{Q_T} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n)dxdt \ge \epsilon(n,\mu,\eta,\eta,j) \tag{78}$$

$m \ge 1$.

Démonstration: We adopt the same technics in the proof in [8].

Proof of (72): If we take $n > m+1$, we get $\Phi_n(u_n)\exp(G(u_n))S_m(u_n) =$

$$\Phi(T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)),$$

then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(Q)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain

$$\Phi_n(u_n)\exp(G(u_n))S_m(u_n) \to \Phi(u)\exp(G(u))S_m(u),$$

with the modular convergence as $n \to +\infty$, then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n) \to \Phi(u)\exp(G(u))S_m(u)$ for $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$.

On the other hand $\nabla W^{n,j}_{\mu,\eta} = \nabla T_k(u_n) - \nabla(T_k(v_j))_\mu$ for $|T_k(u_n) - (T_k(v_j))_\mu| \le \eta$ converge to $\nabla T_k(u) - \nabla(T_k(v_j))_\mu$ weakly in $(L_M(Q_T))^N$, then $\int_{Q_T} \Phi_n(u_n)\exp(G(u_n))S_m(u_n)\nabla W^{n,j}_{\mu,\eta}\,dx\,dt \to \int_{Q_T} \Phi(u)S_m(u)\exp(G(u))\nabla W^{j}_{\mu,\eta}\,dx\,dt$, as $n \to +\infty$.

using the modular convergence of $W^{j}_{\mu,\eta}$ as $j \to +\infty$ and letting μ tends to infinity, we get (72).

Proof of (73): For $n > m+1 > k$, we have $\nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n)$ a.e. in Q_T. By the almost every where convergence of u_n we have $\exp(G(u_n))W^{n,j}_{\mu,\eta} \to \exp(G(u))W^{j}_{\mu,\eta}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\Phi_n(T_{m+1}(u_n)))_n$ converge strongly in $E_{\overline{M}}(Q_T)$ then $\Phi_n(T_{m+1}(u_n))\exp(G(u_n))\ W^{n,j}_{\mu,\eta} \to \Phi(x,t,T_{m+1}(u))\ \exp(G(u))W^{j}_{\mu,\eta}$, converge strongly in $E_{\overline{M}}(Q_T)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(T_{m+1}(u_n))\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$\to \int_{\{m\le|u|\le m+1\}} \Phi(u)\nabla u \exp(G(u))W^{j}_{\mu,\eta}\,dx\,dt$$

as $n \to +\infty$.

with the modular convergence of $W^{j}_{\mu,\eta}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (73).

Proof of (74): we have

$$\int_{Q_T} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n \times\exp(G(u_n))\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$= \int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n \times\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt$$

$$\le \eta C \int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n\,dx\,dt.$$

Using (60), we get

$$\int_{Q_T} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n \exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,ds \le \epsilon(n,\mu,m).$$

Proof of (76): Since $S_m(r) \leq 1$ and $W^{n,j}_{\mu,\eta} \leq \eta$ we get

$$\int_{Q_T} f_n S_m(u_n) \exp(G(u_n)) W^{n,j}_{\mu,\eta} \, dx\,dt \leq \epsilon(n,\eta),$$

$$\int_{Q_T} h(x,t) \exp(G(u_n)) W^{n,j}_{\mu,\eta} S_m(u_n) \, dx\,dt \leq C\eta.$$

Proof of (75):

$$\int_{Q_T} a_n(u_n, \nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla W^{n,j}_{\mu,\eta} \, dx\,dt$$
$$= \int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} a_n(T_k(u_n), \nabla T_k(u_n))$$
$$\times S_m(u_n) \exp(G(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)_\mu) \, dx\,dt$$
$$- \int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} a_n(u_n, \nabla u_n)$$
$$\times S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j)_\mu \, dx\,dt \qquad (79)$$

Since $a_n(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$ there exist some $\varpi_{k+\eta} \in (L_{\overline{M}}(Q_T))^N$ such that $a_n(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \rightarrow \varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(Q_T))^N$.
Consequently,

$$\int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} a_n(u_n, \nabla u_n) S_m(u_n)$$
$$\exp(G(u_n)) \nabla T_k(v_j)_\mu \, dx\,dt$$
$$= \int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)_\mu\leq\eta\}} \varpi_{k+\eta}$$
$$\times S_m(u) \exp(G(u)) \nabla T_k(v_j)_\mu \, dx\,dt + \epsilon(n), \qquad (80)$$

where we have used the fact that
$S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j)_\mu) \chi_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}$

$\rightarrow S_m(u) \exp(G(u)) \nabla T_k(v_j)_\mu) \chi_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)_\mu\leq\eta\}}$

strongly in $(E_M(Q_T))^N$.
Letting $j \rightarrow +\infty$, we obtain

$$\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)_\mu\leq\eta\}} \varpi_{k+\eta}$$
$$S_m(u) \exp(G(u)) \nabla T_k(v_j)_\mu \, dx\,dt$$
$$= \int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(u)_\mu\leq\eta\}} \varpi_{k+\eta}$$
$$S_m(u) \exp(G(u)) \nabla T_k(u)_\mu \, dx\,dt + \epsilon(n,j),$$

One easily has,
$\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(u)_\mu\leq\eta\}} S_m(u) \exp(G(u)) \nabla T_k(u)_\mu \varpi_{k+\eta} \, dx\,dt$

$$= \epsilon(n,j,\mu).$$

By (70)-(76), (79) and (80) we obtain

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} a_n(T_k(u_n), \nabla T_k(u_n)) S_m(u_n)$$
$$\exp(G(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)_\mu) \, dx\,dt$$
$$\leq C\eta + \epsilon(n,j,\mu,m),$$

we know that $\exp(G(u_n)) \geq 1$ and $S_m(u_n) = 1$ for $|u_n| \leq k$ then

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} a_n(x,t,T_k(u_n), \nabla T_k(u_n))$$
$$\times (\nabla T_k(u_n) - \nabla T_k(v_j)_\mu) \, dx\,dt \leq C\eta + \epsilon(n,j,\mu,m). \qquad (81)$$

Proof of (77): Setting for $s > 0$, $Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \leq s\}$ and $Q^s_j = \{(x,t) \in Q : |\nabla T_k(v_j)| \leq s\}$ and denoting by χ^s and χ^s_j the characteristic functions of Q^s and Q^s_j respectively, we deduce that letting $0 < \delta < 1$, define

$$\Theta_{n,k} = (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))$$
$$\times (\nabla T_k(u_n) - \nabla T_k(u))$$

For $s > 0$, we have
$0 \leq \int_{Q^s} \Theta^\delta_{n,k} \, dx\,dt = \int_{Q^s} \Theta^\delta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} \, dx\,dt$

$$+ \int_{Q^s} \Theta^\delta_{n,k} \chi_{\{T_k(u_n)-T_k(v_j)_\mu>\eta\}} \, dx\,dt$$

The first term of the right-side hand, with the Hölder inequality,
$\int_{Q^s} \Theta^\delta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} \, dx\,dt \leq$

$$\left(\int_{Q^s} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} \, dx\,dt\right)^\delta \left(\int_{Q^s} dx\,dt\right)^{1-\delta}$$
$$\leq C_1 \left(\int_{Q^s} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} \, dx\,dt\right)^\delta$$

Also using the Hölder inequality, the second term of the right-side hand is

$$\int_{Q^s} \Theta^\delta_{n,k} \chi_{\{T_k(u_n)-T_k(v_j)_\mu\eta\}} \, dx\,dt \leq \left(\int_{Q^s} \Theta_{n,k} \, dx\,dt\right)^\delta$$
$$\times \left(\int_{\{T_k(u_n)-T_k(v_j)_\mu>\eta\}} dx\,dt\right)^{1-\delta}$$

since $a(x,t,T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_M(Q_T))^N$ then $\int_{Q^s} \Theta^\delta_{n,k} \chi_{\{T_k(u_n)-T_k(v_j)_\mu\eta\}} \, dx\,dt \leq C_2 meas\{(x,t) \in Q_T : |T_k(u_n) - T_k(v_j)_\mu| > \eta\}^{1-\delta}$
We obtain,

$$\int_{Q^s} \Theta^\delta_{n,k} \, dx\,dt \leq C_1 \left(\int_{Q^s} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} \, dx\,dt\right)^\delta$$
$$+ C_2 meas\{(x,t) \in Q_T : T_k(u_n) - T_k(v_j)_\mu > \eta\}^{1-\delta}$$

On the other hand,

$$\int_{Q^s} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}} \, dx\,dt$$
$$\leq \int_{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta} (a(T_k(u_n), \nabla T_k(u_n))$$
$$- a(T_k(u_n), \nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx\,dt$$

For each $s > r$, $r > 0$, one has

$$0 \le \int_{Q^r \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt$$

$$\le \int_{Q^s \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\, dx\, dt$$

$$= \int_{Q^s \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\, dx\, dt$$

$$\le \int_{Q \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\, dx\, dt$$

$$= \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(v_j)\chi_j^s))(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\, dx\, dt$$

$$+ \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \times (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\, dx\, dt$$

$$+ \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(v_j)\chi_j^s) - a(T_k(u_n), \nabla T_k(u)\chi^s))\nabla T_k(u_n)\, dx\, dt$$

$$- \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(v_j)\chi_j^s) \times \nabla T_k(v_j)\chi_j^s)\, dx\, dt$$

$$+ \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s)\, dx\, dt$$

$$= I_1(n,j,s) + I_2(n,j) + I_3(n,j) + I_4(n,j,\mu) + I_5(n,\mu)$$

we will go to the limit as n, j, μ, and $s \to +\infty$ $I_1 = \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n))$

$$\times (\nabla T_k(u_n) - \nabla T_k(v_j)_\mu)\, dx\, dt$$

$$- \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \times (\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\, dx\, dt$$

$$- \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(v_j)\chi_j^s)) \times (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s))\, dx\, dt$$

Using (81), the first term of the right-hand side, we get $\int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)_\mu)\, dx\, dt$

$$\le C\eta + \epsilon(n,m,j,s)$$

$$- \int_{\{|u|>k\} \cap \{|T_k(u) - T_k(v_j)_\mu| \le \eta\}} a(T_k(u), 0)\nabla T_k(v_j)_\mu\, dx\, dt \le C\eta + \epsilon(n,m,j,\mu)$$

The second term of the right-hand side tends to

$$\int_{\{|T_k(u) - T_k(v_j)_\mu| \le \eta\}} \varpi_k(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\, dx\, dt$$

since $a(T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, there exist some $\varpi_k \in (L_{\overline{M}}(Q_T))^N$ such that (for a subsequence still denoted by u_n

$$a(T_k(u_n), \nabla T_k(u_n)) \to \varpi_k \quad \text{in} \quad (L_M(Q_T))^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M)$$

In view of the fact that $(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\chi_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} \to (\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\chi_{\{0 \le T_k(u) - T_k(v_j)_\mu \le \eta\}}$ Strongly in $(E_M(Q_T))^N$ as $n \to +\infty$.

the third term of the right-hand side tends to $\int_{\{0 \le T_k(u) - T_k(v_j)_\mu \le \eta\}} a(T_k(u), \nabla T_k(v_j)\chi_j^s))$

$$(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s))\, dx\, dt$$

Since $a(T_k(u_n), \nabla T_k(v_j)\chi_j^s))\chi_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} \to a(T_k(u), \nabla T_k(v_j)\chi_j^s))\chi_{|T_k(u) - T_k(v_j)_\mu| \le \eta}$ in $(E_{\overline{M}}(Q_T))^N$ while

$$(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)) \to (\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s))$$

in $(L_M(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_1 \le C\eta + \epsilon(n,j,s,\mu)$$

For what concerns I_2, by letting $n \to +\infty$, we have

$$I_2 \to \int_{\{0 \le T_k(u) - T_k(v_j)_\mu \le \eta\}} \varpi_k(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\, dx\, dt$$

Since $a(T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k$ in $(L_{\overline{M}}(Q_T))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$, while $(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}}$

$$\to (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{\{0 \le T_k(u) - T_k(v_j)_\mu| \le \eta\}}$$

strongly in $(E_M(Q_T))^N$.

Passing to limit $j \to +\infty$, and using Lebesgue's theorem, we have

$$I_2 = \epsilon(n,j)$$

Similar ways as above give

$$I_3 = \epsilon(n,j)$$

$$I_4 = \int_{\{0 \le T_k(u) - T_k(u)_\mu \le \eta\}} a(T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx\, dt + \epsilon(n,j,\mu,s,m)$$

$$I_5 = \int_{\{0\le T_k(u)-T_k(u)_\mu\le\eta\}} a(T_k(u),\nabla T_k(u))\nabla T_k(u)\,dx\,dt$$
$$+\epsilon(n,j,\mu,s,m)$$

Finally, we obtain,

$$\int_{Q^s}\Theta_{n,k}\,dx\,dt \le C_1(C\eta+\epsilon(n,\mu,\eta,m))^\delta + C_2(\epsilon(n,\mu,))^{1-\delta}$$

Which yields,by passing to the limit sup over n,j,μ ,s and η

$$\int_{Q^r\cap\{0\le T_\eta(T_k(u)-T_k(u)_\mu)\}}[(a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)))$$
$$(\nabla T_k(u_n)-\nabla T_k(u))]^\delta\,dx\,dt=\epsilon(n) \quad (82)$$

Taking on the hand the function $W^{n,j}_{\eta,\mu} = T_\eta(T_k(u_n) - (T_k(v_j))_\mu)^-$ and $W^{j}_{\eta,\mu} = T_\eta(T_k(u) - (T_k(v_j))_\mu)^-$.

Multiplying the approximating equation by $\exp(G(u_n)))W^{n,j}_{\eta,\mu}S_m(u_n)$, we obtain

$$\int_{Q^r\cap\{T_\eta(T_k(u_n)-(T_k(v_j))_\mu)\le 0\}}[(a(x,T_k(u_n),\nabla T_k(u_n))$$
$$-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))]^\delta\,dxdt=\epsilon(n) \quad (83)$$

by (82) and (83) we get

$$\int_{Q^r}[(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(u))]^\delta\,dxdt=\epsilon(n)$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Q^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T$$

Step 4: Equi-integrability of the nonlinearity sequence

We shall prove that $H_n(u_n,\nabla u_n) \to H(u,\nabla u)$ strongly in $L^1(Q_T)$.

Consider $g_0(u_n) = \int_0^{u_n}\rho(s)\chi_{\{s>h\}}ds$ and multiply (26) by $\exp(G(u_n))g_0(u_n)$, we get

$$\Big[\int_{Q_T}B^n_h(x,u_n)dx\Big]_0^T+\int_{Q_T}a(,u_n,\nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt$$
$$+\int_{Q_T}\Phi_n(u_n,\nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt$$
$$+\int_{Q_T}H_n(u_n,\nabla u_n)\exp(G(u_n))g_0(u_n))dxdt$$

$$\le\Big(\int_h^{+\infty}\rho(s)dx\Big)\exp\Big(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\Big)[\|f_n\|_{L^1(Q)}+\|h(x,t\|_{L^1(Q_T)}]$$

where $B^n_h(x,r)=\int_0^r\frac{\partial b(x,s)}{\partial s}g_0(s)\exp(G(T_k(s)))ds\ge 0$ then using same technique in step 2 we can have

$$\int_{\{u_n>h\}}\rho(u_n)M(x,\nabla u_n)dxdt\le C\Big(\int_h^{+\infty}\rho(s)dx\Big)$$

Since $\rho\in L^1(I\!R)$, we get

$$\lim_{h\to 0}\sup_{n\in I\!N}\int_{\{u_n>h\}}\rho(u_n)M(x,\nabla u_n)dxdt=0$$

Similarly, let $g_0(u_n)=\int_{u_n}^0\rho(s)\chi_{\{s<-h\}}dx$ in (26), we have also

$$\lim_{h\to 0}\sup_{n\in I\!N}\int_{\{u_n<-h\}}\rho(u_n)M(x,\nabla u_n)dxdt=0$$

We conclude that

$$\lim_{h\to 0}\sup_{n\in I\!N}\int_{\{|u_n|>h\}}\rho(u_n)M(x,\nabla u_n)dxdt=0 \quad (84)$$

Let $D\subset Q_T$ then

$$\int_D\rho(u_n)M(x,\nabla u_n)dxdt\le\max_{\{|u_n|\le h\}}\rho(y)\int_{D\cap\{|u_n|\le h\}}M(x,\nabla u_n)dxdt$$
$$+\int_{D\cap\{|u_n|>h\}}\rho(u_n)M(x,\nabla u_n)dxdt$$

Consequently $\rho(u_n)M(x,\nabla u_n)$ is equi-integrable. Then $\rho(u_n)M(x,\nabla u_n)$ converge to $\rho(u)M(x,\nabla u)$ strongly in $L^1(I\!R)$. By (16), we get our result.

As a conclusion, the proof of Theorem (4) is complete.

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