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Doubly Nonlinear Parabolic Systems In Inhomogeneous Musielak-Orlicz-Sobolev Spcaes

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ABSTRACT

In this paper, we discuss the solvability of the nonlinear parabolic systems associated to the nonlinear parabolic equation: $\frac{\partial b_i(x,u_i)}{\partial t} - div(a(x,t,u_i,\nabla u_i)) - \phi_i(x,t,u_i)) + f_i(x,u_1,u_2) = 0$, *where the function* $b_i(x,u_i)$ *verifies some regularity conditions, the term* $\Big(a(x,t,u_i,\nabla u_i)\Big)$ *is a generalized Leray-Lions operator and* ϕ_i *is a Carathéodory function assumed to be continuous on* u_i *and satisfy only a growth condition. The source term* $f_i(t,u_1,u_2)$ *belongs to* $L^1(\Omega\times(0,T))$.

1 Introduction

Given a bounded-connected open set $\Omega$of $\mathbb{R}^N$ $(N=2)$, with Lipschitz boundary $\partial\Omega$, $Q_T=\Omega\times(0,T)$ is the generic cylinder of an arbitrary finite hight, $T<+\infty$. We prove the existence of a renormalized solutions for the nonlinear parabolic systems:

$$\frac{\partial b_i(x,u_i)}{\partial t} - \mathrm{div}(a(x,t,u_i,\nabla u_i)) - div(\Phi_i(x,t,u_i)) + f_i(x,u_1,u_2) = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$u_i = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.2}$$

$$b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) \quad \text{in } \Omega, \tag{1.3}$$

where i=1,2. Here the vector field $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function such that $-\mathrm{div}(a(x,t,u_i,\nabla u_i))$ is a Leray-lions operator defined from the Inhomogeneous Musielak-Orlicz-Sobolev Spcaes $W_0^{1,x}L_\varphi(Q_T)$into its dual $W^{-1,x}L_\psi(Q_T)$. Let $b_i:\Omega\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function such that for every $x\in\Omega$, $b_i(x,.)$ is a strictly increasing C^1-function, the divegential term $\Phi_i(x,t,u_i)$ is a Carathéodory function satisfy only a polynomial growth with respect to the anisotropic N-function φ (see (4.6)), the data $u_{0,i}$ is in $L^1(\Omega)$ such that $b_i(.,u_{0,i})$ in $L^1(\Omega)$ and the source f_i is a Carathéodory function satisfy the assumptions ((4.7)-(4.10)). When problem ((1.1)) is investigated, there is a difficulty due to the fact that the data $b_1(x,u_0^1(x))$ and $b_2(x,u_0^2(x))$ only belong to L^1 and the functions $a(x,t,u_i,\nabla u_i)$,$\Phi_i(x,t,u_i)$ and $f_i(x,u_1,u_2)$ do not belong to $(L^1_{loc}(Q_T))^N$ in general, so that proving existence of weak solution seems to be an arduous task, and we cannot use the Stocks formula in the a priori estimates of the nonlinearity ,$\Phi_i(x,t,u_i)$. In order to overcome this difficulty, we work with the framework of renormalized solutions (see Definition 3.1). One of the models of applications of these operators is the system of Boussinesq:

$$\frac{\partial u}{\partial t} + (u.\nabla))u - 2div(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) \quad \text{in} \quad Q_T$$

$$\frac{\partial b(\theta)}{\partial t} + u.\nabla b(\theta) - \triangle\theta = 2\mu(\theta)|\varepsilon(u)|^2 \quad \text{in} \quad Q_T$$

$$u(t=0)=u_0,\; b(\theta)(t=0)=b(\theta_0) \quad \text{on} \quad \Omega$$

$$u=0 \quad \theta=0 \quad \text{on} \quad \partial\Omega\times(0,T)$$

Equation first equation is the motion conservation equation, the unknowns are the fields of displacement $u:Q_T\to\mathbb{R}^N$ and temperature $\theta:Q_T\to\mathbb{R}$, The field $\varepsilon(u)=\frac{1}{2}(\nabla u+(\nabla u)^t)$ is the strain rate tensor.

It is our purpose, in this paper to generalize the result of ([1], [2], [3]) and we prove the existence of a renormalized solution of system (1.1).

The plan of the paper is as follows: In Section 2 we give the framework space, in Section 3 and 4 we

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give some useful Lemmas and basics assumptions. In Section 5 we give the definition of a renormalized solution of (1.1), and we establish (Theorem 5.1) the existence of such a solution.

2 Preliminaries

2.1 Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ ($N \geq 2$), and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying conditions:

Φ_1:$\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $\varphi(x,0) = 0$,$\varphi(x,0) > 0$ for $t > 0$, $\lim_{t\to 0}\sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty}\inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$).

Φ_2:$\varphi(.,t)$ is a measurable function for all $t \geq 0$.

A function φ which satisfies the conditions Φ_1 and Φ_2 is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$$

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp.globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$
$\gamma(x,t) \leq \varphi(x,ct)$ for all $t \geq t_0$ (resp. for all $t \geq 0$). We say that γ grows essentially less rapidly than φ at 0(resp. near infinity) and we write $\gamma << \varphi$, for every positive constant c, we have
$\lim_{t\to 0}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0$
(resp.$\lim_{t\to\infty}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0$)

Remark 2.1 *[4]. If $\gamma << \varphi$ near infinity,then $\forall \epsilon > 0$ there exist $k(\epsilon) > 0$ such that for almost all $x \in \Omega$, we have*

$$\gamma(x,t) \leq k(\epsilon)\varphi(x,\epsilon t) \quad \forall t \geq 0$$

2.2 Musielak-Orlicz space

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \to \mathbb{R}$, we define the functionnal

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R}$ mesurable : $\rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class . The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$; that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable :} \quad \rho_{\varphi,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}$$

For any Musielak-Orlicz function φ, we put $\psi(x,s) = \sup_{t\geq 0}(st - \varphi(x,s))$.

ψ is called the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to s. We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \rho_{\varphi,\Omega}(\frac{u_n - u}{\lambda}) = 0$

This implies convergence for $\sigma(\Pi L_\varphi, \Pi L_\psi)$[see [5]].

In the space $L_\varphi(\Omega)$, we define the following two norms

$$\|u\|_\varphi = \inf\left\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx \leq 1\right\}$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [8]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$. The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is by denoted $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$, if and only if φ satisfies the Δ_2−condition for large values of t or for all values of t, according to whether Ω has finite measure or not.

We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall \alpha \leq 1\}$$
$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall \alpha \leq 1\}$$

where $\alpha = (\alpha_1, ..., \alpha_N)$,$|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and : $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.
Let
$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \rho_{\varphi,\Omega}(D^\alpha u)$ and $\|u\|^1_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$ for$u \in W^1L_\varphi(\Omega)$.

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively. Then pair $(W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition [6].

$$\text{There exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x\in\Omega} \varphi(x,1) > c$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\prod_{\alpha\leq 1} L_\varphi(\Omega) = \prod L_\varphi$ We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R})$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the(norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$. For two complementary Musielak-Orlicz functions φ and ψ, we have (See [7]).

- The Young inequality:

$$st \leq \varphi(x,s) + \psi(x,t) \text{ for all } s,t \geq 0 \text{ , } x \in \Omega.$$

- The Hölder inequality

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \|u\|_{\varphi,\Omega}|||v|||_{\psi,\Omega} \text{ for all } u \in L_\varphi(\Omega), v \in L_\psi(\Omega)$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W_0^1L_\varphi(\Omega)$) if, for some $\lambda > 0$.

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0$$

The following spaces of distributions will also be used

$$W^{-1}L_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\le 1} (-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in L_\psi(\Omega) \Big\}$$

and

$$W^{-1}E_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\le 1} (-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in E_\psi(\Omega) \Big\}$$

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spcaes:

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $Q = \Omega\times]0,T[$ with some given $T > 0$. let φ be a Musielak-Orlicz function.For each $\alpha \in N^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in R^N$. The inhomogeneous Musielak-Orlicz-Sovolev spaces of order 1 are defined as follows

$$W^{1,x}L_\varphi(Q) = \Big\{ u \in L_\varphi(Q) : \forall|\alpha| \le 1, \quad D_x^\alpha u \in L_\varphi(Q) \Big\}$$

$$W^{1,x}E_\varphi(Q) = \Big\{ u \in E_\varphi(Q) : \forall|\alpha| \le 1, \quad D_x^\alpha u \in E_\varphi(Q) \Big\}$$

The last is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\le m} \|D_x^\alpha u\|_{\varphi,Q}$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q)$ then the function : $t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1L_\varphi(\Omega)$. If, further, $u \in W^{1,x}E_\varphi(Q)$ then this function is a $W^1E_\varphi(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_\varphi(Q) \subset L^1(0,T;W^1E_\varphi(\Omega))$. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, if $u \in W^{1,x}L_\varphi(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto u(t) = \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q)$ of $\mathcal{D}(\Omega)$. We can easily show that when Ω is a Lipschitz domain then each element u of the closure of $\mathcal{D}(\Omega)$ with respect of the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit, in $W^{1,x}L_\varphi(Q)$, of some subsequence $(u_i) \in \mathcal{D}(\Omega)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \le 1$,

$$\int_Q \varphi(x, (\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}) dxdt \to 0 \quad \text{as} \quad i \to \infty$$

, this implies that (u_i) converge to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$. Consequently

$$\mathcal{D}(Q)^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \mathcal{D}(Q)^{\sigma(\Pi L_\varphi, \Pi L_\psi)}$$

, this space will be denoted by $W_0^{1,x}L_\varphi(Q)$. Furthermore $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi$. We have the following complementary system F being the dual space of $W_0^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, and will be denoted by $F = W^{1,x}L_\psi(Q)$ and it is shown that

this space will be equipped with the usual quotient norm

where the inf is taken on all possible decompositions

The space F_0 is then given by $F_0 = W^{-1,x}E_\psi(Q)$.

Lemma 2.1 *[4]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

- *There exists a constant $c > 0$ such that*

$$\inf_{x\in\Omega} \varphi(x,1) > c \tag{2.1}$$

- $\exists A > 0$ *such that for all* $x, y \in \Omega$ *with* $|x - y| \le \frac{1}{2}$, *we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \le t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \ge 1. \tag{2.2}$$

-

$$\int_\Omega \varphi(y,1)dx < \infty \tag{2.3}$$

-

$$\exists C > 0 \quad \text{such that} \quad \psi(y,t) \le C \quad a.e. \quad in \quad \Omega \tag{2.4}$$

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

2.4 Truncation Operator

T_k, $k > 0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2.2 *[4]. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian,with $F(0) = 0$. Let φ be an Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$(resp.$u \in W^1 E_\varphi(\Omega)$). Then $F(u) \in W^1 L_\varphi(\Omega)$(resp.$u \in W_0^1 E_\varphi(\Omega)$).Moreover, if the set of discontinuity points D of F' is finite,then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega;\, u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega;\, u(x) \in D\} \end{cases}$$

Lemma 2.3 *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying :

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty \quad \text{a.e.} \quad x \in \Omega \tag{2.5}$$

and the conditions of Lemma 2.1. We may assume without loss of generality that

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty \quad \text{a.e.} \quad x \in \Omega \tag{2.6}$$

Define a function $\varphi^* : \Omega \times [0, \infty)$ by $\varphi^*(x, s) = \int_0^s \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt$ $x \in \Omega$ and $s \in [0, \infty)$.

φ^*its called the Sobolev conjugate function of φ (see [1] for the case of Orlicz function).

Theorem 2.1 *Let Ω be a bounded Lipschitz domain and let φ be a Musielak-Orlicz function satisfying 2.5,2.6 and the conditions of Lemma 2.1. Then*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_{\varphi^*}(\Omega)$$

where φ^* is the Sobolev conjugate function of φ. Moreover, if ϕ is any Musielak-Orlicz function increasing essentially more slowly than φ^* near infinity, then the imbedding

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_\phi(\Omega)$$

is compact

Corollary 2.1 *Under the same assumptions of theorem 5.1, we have*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow\hookrightarrow L_\varphi(\Omega)$$

Lemma 2.4 *If a sequence un $u_n \in L_\varphi(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_\varphi(\Omega)$, then $u \in L_\varphi(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.*

Lemma 2.5 *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

See ([8]).

3 Technical lemma

Lemma 3.1 *Under the assumptions of lemma 2.1, and by assuming that $\varphi(x,t)$ decreases with respect to one of coordinate of x, there exists a constant $c_1 > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x, |u|) dx \leq \int_\Omega \varphi(x, c_1 |\nabla u|) dx \tag{3.1}$$

Theorem 3.1 *Let Ω be a bounded Lipschitz domain and let φ be a Musielak-Orlicz function satisfying the same conditions of Theorem 5.1. Then there exists a constant $\lambda > 0$ such that*

$$\|u\|_\varphi \leq \lambda \|\nabla u\|_\varphi, \quad \forall \in W_0^1 L_\varphi(\Omega)$$

4 Essential assumptions

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$ satisfying the segment property,and let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfies conditions of Lemma 2.1 and $\gamma << \varphi$.

$$b_i : \Omega \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function such that for every $x \in \Omega$, (4.1)

$b_i(x,.)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function and $b_i \in L^\infty(\Omega \times \mathbb{R})$ with $b_i(x, 0) = 0$. Next for any $k > 0$, there exists a constant $\lambda_k^i > 0$ and functions $A_k^i \in L^\infty(\Omega)$ and $B_k^i \in L_\varphi(\Omega)$ such that:

$$\lambda_k^i \leq \frac{\partial b_i(x,s)}{\partial s} \leq A_k^i(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b_i(x,s)}{\partial s}\right)\right| \leq B_k^i(x)$$

$$\text{a.e. } x \in \Omega \text{ and } \forall\ |s| \leq k. \tag{4.2}$$

$A : D(A) \subset W_0^1 L_\varphi(Q_T) \to W^{-1} L_\psi(Q_T)$ defined by $A(u) = -diva(x,t,u,\nabla u)$,where $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$

(A_1) : $|a(x,t,s,\xi)| \leq \beta(c(x) + \psi_x^{-1}(\gamma(x, \nu_1|s|)) + \psi_x^{-1}(\varphi(x, \nu_2|\xi|)))$,

$$\beta > 0, \quad c(x) \in E_\psi(\Omega), \tag{4.3}$$

$$(A_2) : (a(x,t,s,\xi) - a(x,s,\xi^*)(\xi - \xi^*) > 0, \tag{4.4}$$

$$(A_3) : a(x,t,s,\xi).\xi \geq \alpha\varphi(x, |\xi|). \tag{4.5}$$

$\Phi(x,s,\xi) : \Omega \times IR \times IR^N \to IR^N$ is a Carathéodory function such that

$$|\Phi_i(x,t,s)| \leq \psi_x^{-1}\varphi(x,|s|), \tag{4.6}$$

$f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with

$$f_1(x, 0, s) = f_2(x, s, 0) = 0 \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}. \tag{4.7}$$

and for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,

$$sign(s_i) f_i(x, s_1, s_2) \geq 0. \tag{4.8}$$

The growth assumptions on f_i are as follows: For each $K > 0$, there exists $\sigma_K > 0$ and a function F_K in $L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \leq F_K(x) + \sigma_K \, |b_2(x, s_2)| \tag{4.9}$$

a.e. in Ω, for all s_1 such that $|s_1| \leq K$, for all $s_2 \in \mathbb{R}$. For each $K > 0$, there exists $\lambda_K > 0$ and a function G_K in $L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \leq G_K(x) + \lambda_K \, |b_1(x, s_1)|, \tag{4.10}$$

for almost every $x \in \Omega$, for every s_2 such that $|s_2| \leq K$, and for every $s_1 \in \mathbb{R}$.
Finally, we assume the following condition on the initial data $u_{i,0}$: for i=1,2.

$$u_{i,0} \text{ is a measurable function such that } b_i(., u_{i,0}) \in L^1(\Omega), \tag{4.11}$$

In this paper, for $K > 0$, we denote by $T_K : r \mapsto \min(K, max(r, -K))$ the truncation function at height K. For any measurable subset E of Q_T, we denote by $meas(E)$ the Lebesgue measure of E. For any measurable function v defined on Q and for any real number $s, \chi_{\{v<s\}}$ (respectively, $\chi_{\{v=s\}}, \chi_{\{v>s\}}$) denote the characteristic function of the set $\{(x,t) \in Q_T \; ; \; v(x,t) < s\}$ (respectively, $\{(x,t) \in Q_T ; v(x,t) = s\}, \{(x,t) \in Q_T ; v(x,t) > s\}$).

Definition 4.1 *A couple of functions* (u_1, u_2) *defined on* Q *is called a renormalized solution of* (4.1)-(4.11)*if for* $i = 1, 2$ *the function* u_i *satisfies*

$$T_K(u_i) \in W_0^{1,x} L_\varphi(Q_T) \quad and \quad b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)), \tag{4.12}$$

$$\int_{\{ m \leq |u_i| \leq m+1\}} a(x, t, u_i, \nabla u_i) \nabla u_i \, dx \, dt \to 0 \quad as \; m \to +\infty, \tag{4.13}$$

For every function S *in* $W^{2,\infty}(\mathbb{R})$ *which is piecewise* C^1 *and such that* S' *has a compact support,we have*

$$\frac{\partial B_{i,S}(x, u_i)}{\partial t} - div(S'(u_i) a(x, t, u_i, \nabla u_i))$$
$$+ S''(u_i) a(x, t, u_i, \nabla u_i) \nabla u_i$$

$$+ div(S'(u_i) \phi_i(x, t, u_i)) - S''(u_i) \phi_i(x, t, u_i) \nabla u_i$$
$$+ f_i(x, u_1, u_2) S'(u_i) = 0, \tag{4.14}$$

$$B_{i,S}(x, u_i)(t = 0) = B_{i,S}(x, u_{i,0}) \quad in \; \Omega, \tag{4.15}$$

where $B_{i,S}(r) = \int_0^r b_i'(x, s) S'(s) \, ds$.

Due to (4.12), each term in (4.14) has a meaning in $W^{-1,x} L_\psi(Q_T) + L^1(Q_T)$.
Indeed, if K such that $suppS \subset [-K, K]$, the following identifications are made in (4.14)

- $B_{i,S}(x, u_i) \in L^\infty(Q_T)$, since $|B_{i,S}(x, u_i)| \leq K \|A_K^i\|_{L^\infty(\Omega)} \|S'\|_{L^\infty(\mathbb{R})}$
- $S'(u_i) a(x, t, u_i, \nabla u_i)$ can be identified with $S'(u_i) a(x, t, T_K(u_i), \nabla T_K(u_i))$ a.e. in Q_T. Since indeed $|T_K(u_i)| \leq K$ a.e. in Q_T, . As a consequence of (4.3) , (4.12) and $S'(u_i) \in L^\infty(Q_T)$, it follows that
$$S'(u_i) a(x, T_K(u_i), \nabla T_K(u_i)) \in (L_\psi(Q_T))^N.$$
- $S'(u_i) a(x, t, u_i, \nabla u_i) \nabla u_i$ can be identified with $S'(u_i) a(x, t, T_K(u_i), \nabla T_K(u_i)) \nabla T_K(u_i)$ a.e. in Q_T.with (4.2) and (4.12) it has
$$S'(u_i) a(x, t, T_K(u_i), \nabla T_K(u_i)) \nabla T_K(u_i) \in L^1(Q_T)$$
- $S'(u_i) \Phi_i(u_i)$ and $S''(u_i) \Phi_i(u_i) \nabla u_i$ respectively identify with $S'(u_i) \Phi_i(T_K(u_i))$ and $S''(u_i) \Phi(T_K(u_i)) \nabla T_K(u_i)$. In view of the properties of S and (4.6), the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (4.12) implies that $S'(u_i) \Phi_i(T_K(u_i)) \in (L^\infty(Q_T))^N$ and $S''(u_i) \Phi_i(T_K(u_i)) \nabla T_K(u_i) \in (L_\psi(Q_T))^N$.
- $S'(u_i) f_i(x, u_1, u_2)$ identifies with $S'(u_i) f_1(x, T_K(u_1), u_2)$ a.e. in Q_T
(or $S'(u_i) f_2(x, u_1, T_K(u_2))$ a.e. in Q_T). Indeed, since $|T_K(u_i)| \leq K$ a.e. in Q_T, assumptions (4.9) and (4.10) and using (4.12) and of $S'(u_i) \in L^\infty(Q)$, one has
$$S'(u_1) f_1(x, T_K(u_1), u_2) \in L^1(Q_T)$$
$$\text{and} \quad S'(u_2) f_2(x, u_1, T_K(u_2)) \in L^1(Q_T).$$

As consequence, (4.14) takes place in $D'(Q_T)$ and that

$$\frac{\partial B_{i,S}(x, u_i)}{\partial t} \in W^{-1,x} L_\psi(Q_T) + L^1(Q_T). \tag{4.16}$$

Due to the properties of S and (4.2)

$$B_{i,S}(x, u_i) \in W_0^{1,x} L_\varphi(Q_T). \tag{4.17}$$

Moreover (4.16) and (4.17) implies that $B_{i,S}(x, u_i) \in C^0([0, T], L^1(\Omega))$ so that the initial condition (4.15) makes sense.

5 Existence result

We shall prove the following existence theorem

Theorem 5.1 *Assume that* (4.1)-(4.11) *hold true. There at least a renormalized solution* (u_1, u_2) *of Problem* (1.1).

We divide the prof in 5 steps.
Step 1: Approximate problem.
Let us introduce the following regularization of the data: for $n > 0$ and $i = 1, 2$

$$b_{i,n}(x, s) = b_i(x, T_n(s)) + \frac{1}{n} s \quad \forall s \in \mathbb{R}, \tag{5.1}$$

$$a_n(x, t, s, \xi) = a(x, t, T_n(s), \xi) \text{ a.e. in } \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{5.2}$$

$$\Phi_{i,n}(x,t,s) = \Phi_{i,n}(x,t,T_n(s)) \quad \text{a.e.} \quad (x,t) \in Q_T, \quad \forall s \in IR. \tag{5.3}$$

$$f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),s_2) \quad \text{a.e. in } \Omega, \forall s_1,s_2 \in \mathbb{R}, \tag{5.4}$$

$$f_{2,n}(x,s_1,s_2) = f_2(x,s_1,T_n(s_2)) \quad \text{a.e. in } \Omega, \forall s_1,s_2 \in \mathbb{R}, \tag{5.5}$$

$$u_{i,0n} \in C_0^\infty(\Omega), b_{i,n}(x,u_{i,0n}) \to b_i(x,u_{i,0}) \quad \text{in} L^1(\Omega)$$

$$\text{as } n \text{ tends to } +\infty \tag{5.6}$$

Let us now consider the regularized problem $\frac{\partial b_{i,n}(x,u_{i,n})}{\partial t} - \text{div}(a_n(x,u_{i,n},\nabla u_{i,n})) - div(\Phi_{i,n}(x,t,u_{i,n}))$

$$+ f_{i,n}(x,u_{1,n},u_{2,n}) = 0 \quad \text{in } Q_T, \tag{5.7}$$

$$u_{i,n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{5.8}$$

$$b_{i,n}(x,u_{i,n})(t=0) = b_{i,n}(x,u_{i,0n}) \quad \text{in } \Omega. \tag{5.9}$$

In view of (5.1), for $i = 1,2$, we have

$$\frac{\partial b_{i,n}(x,s)}{\partial s} \geq \frac{1}{n}, \quad |b_{i,n}(x,s)| \leq \max_{|s|\leq n}|b_i(x,s)| + 1 \quad \forall s \in \mathbb{R},$$

In view of (4.9)-(4.10), $f_{1,n}$ and $f_{2,n}$ satisfy: There exists $F_n \in L^1(\Omega), G_n \in L^1(\Omega)$ and $\sigma_n > 0, \lambda_n > 0$, such that

$|f_{1,n}(x,s_1,s_2)| \leq F_n(x) + \sigma_n \max_{|s|\leq n}|b_i(x,s)|$ a.e. in $x \in \Omega, \forall s_1,s_2 \in \mathbb{R}$,

$|f_{2,n}(x,s_1,s_2)| \leq G_n(x) + \lambda_n \max_{|s|\leq n}|b_i(x,s)|$ a.e. in $x \in \Omega, \forall s_1,s_2 \in \mathbb{R}$. As a consequence, proving the existence of a weak solution $u_{i,n} \in W_0^{1,x}L_\varphi(Q_T)$ of (5.7)-(5.9) is an easy task (see e.g. [9]).

Step2:A priori estimates.

Let $t \in (0,T)$ and using $T_k(u_{i,n})\chi_{(0,t)}$ as a test function in problem (5.7), we get:

$$\int_\Omega B_{i,k}^n(x,u_{i,n}(t))dx + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})dx\,dt$$

$$+ \int_{Q_t} \phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt \tag{5.10}$$

$$+ \int_{Q_t} f_{i,n} T_k(u_{i,n})\,dx\,dt \leq \int_\Omega B_{i,k}^n(x,u_{i,0n})dx,$$

where $B_{i,k}^n(x,r) = \int_0^r \frac{\partial b_{i,n}(x,s)}{\partial s} T_k(s)ds$.

Due to definition of $B_{i,k}^n$ we have:

$$\int_\Omega B_{i,k}^n(x,u_{i,n}(t))dx \geq \frac{\lambda_n}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx, \quad \forall k > 0, \tag{5.11}$$

and

$$0 \leq \int_\Omega B_{i,k}^n(x,u_{i,0n})dx \leq k\int_\Omega |b_{i,n}(x,u_{i,0n})|dx \tag{5.12}$$

$$\leq k\|b_i(x,u_{i,0})\|_{L^1(\Omega)}, \quad \forall k > 0.$$

In view of (4.8), we have $\int_{Q_t} f_{i,n}T_k(u_{i,n})\,dx\,dt \geq 0$ Also, we obtain with Young inequality:

$$\int_{Q_t} \phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt$$

$$= \int_{\{|u_{i,n}|\leq k\}} \phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt$$

$$\leq \int_{\{|u_{i,n}|\leq k\}} \psi(x, \frac{1}{\alpha_0^i}\phi_{i,n}(x,t,u_{i,n}))dxdt$$

$$+ \int_{\{|u_{i,n}|\leq k\}} \varphi(x,\alpha_0^i\nabla T_k(u_{i,n}))dxdt$$

$$\leq \int_{\{|u_{i,n}|\leq k\}} \psi(x, \frac{1}{\alpha_0^i}\psi_x^{-1}\varphi(x,|k|))dxdt$$

$$+ \int_{\{|u_{i,n}|\leq k\}} \varphi(x,\alpha_0^i\nabla T_k(u_{i,n})dxdt$$

$$\leq \int_{\{|u_{i,n}|\leq k\}} \psi(x, \frac{1}{\alpha_0^i}\psi_x^{-1}\varphi(x,|k|))dxdt$$

$$+ \int_{\{|u_{i,n}|\leq k\}} \varphi(x,\alpha_0^i\nabla T_k(u_{i,n})dxdt$$

then

$$\int_{Q_t} \phi_{i,n}(x,t,T_k(u_{i,n}))\nabla T_k(u_{i,n})dx\,dt$$

$$\leq C_{i,k} + \alpha_0^i\int_{Q_t}\varphi(x,\nabla T_k(u_{i,n}))dx\,dt \tag{5.13}$$

We conclude that

$$\frac{\lambda}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx + \alpha^i\int_{Q_t}\varphi(x,\nabla T_k(u_{i,n})\,dx\,dt$$

$$\leq \alpha_0^i\int_{Q_t}\varphi(x,\nabla T_k(u_{i,n}))\,dt\,dx + C_{i,k} + k\|b_i(x,u_{i,0n})\|_{L^1(\Omega)}$$

Then

$$\frac{\lambda}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx + (\alpha^i - \alpha_0^i)\int_{Q_t}\varphi(x,\nabla T_k(u_{i,n}))\,dt\,dx \leq C_i.k$$

Choosing α_0^i such that

$$0 < \alpha_0^i < \min(1,\alpha^i)$$

we get

$$\int_{Q_t}\varphi(x,\nabla T_k(u_{i,n}))\,dx\,dt \leq C_i.k \tag{5.14}$$

Then, by (5.14), we conclude that $T_k(u_{i,n})$ is bounded in $W^{1,x}L_\varphi(Q_T)$ independently of n and for any $k \geq 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_{i,n}) \to \psi_{i,k} \tag{5.15}$$

weakly in $W_0^{1,x}L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ strongly in $E_\varphi(Q_T)$ and a.e in Q_T.

Since Lemma(3.1)and (5.14), we get also,

$$\varphi(x,k)\ meas\Big\{\{|u_{i,n}| > k\} \cap B_R \times [0,T]\Big\}$$

$$\leq \int_0^T \int_{\{|u_{i,n}|>k\}\cap B_R} \varphi(x,T_k(u_{i,n}))dxdt$$

$$\leq \int_{Q_T}\varphi(x,T_k(u_{i,n}))dxdt$$

$\leq diamQ_T \int_{Q_T} \varphi(x, \nabla T_k(u_{i,n}))dxdt$

Then

$$meas\Big\{\{|u_{i,n}| > k\} \cap B_R \times [0,T]\Big\} \leq \frac{diamQ_T.C_i.k}{\varphi(x,k)}$$

Which implies that:

$\lim_{k\to+\infty} meas\Big\{\{|u_{i,n}| > k\} \cap B_R \times [0,T]\Big\} = 0.$ uniformly with respect to n.

Now we turn to prove the almost every convergence of $u_{i,n}$, $b_{i,n}(x,u_{i,n})$ and convergence of $a_{i,n}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))$.

Proposition 5.1 *Let $u_{i,n}$ be a solution of the approximate problem, then:*

$$u_{i,n} \to u_i \quad a.e\ in \quad Q_T. \tag{5.16}$$

$$b_{i,n}(x,u_{i,n}) \to b_i(x,u_i) \quad a.e\ in \quad Q_T$$

$$b_i(x,u_i) \in L^\infty(0,T,L^1(\Omega)). \tag{5.17}$$

$$a_n(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) \rightharpoonup X_{i,k}$$

$$in \quad (L_\psi(Q_T))^N for \quad \sigma(\Pi L_\psi, \Pi E_\varphi) \tag{5.18}$$

for some $X_{i,k} \in (L_\psi(Q_T))^N$

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1} a_i(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}dxdt = 0 \tag{5.19}$$

Proof of (5.16) **and** (5.17)**:**

Consider now a function non decreasing $g_k \in C^2(I\!R)$ such that $g_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g_k'(u_{i,n})$, we get

$$\frac{\partial B^{i,n}_{k,g}(x,u_{i,n})}{\partial t} - div\Big(a_n(x,t,u_{i,n},\nabla u_{i,n})g_k'(u_{i,n})\Big)$$
$$+ a_n(x,t,u_{i,n},\nabla u_{i,n})g_k''(u_{i,n})\nabla u_{i,n} \tag{5.20}$$
$$+ div\Big(\phi_{i,n}(x,t,u_{i,n})g_k'(u_{i,n})\Big) - g_k''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}$$
$$+f_{i,n}g_k'(u_n) = 0 \qquad \text{in } D'(Q_T)$$

where $B^{i,n}_{k,g}(x,z) = \displaystyle\int_0^z \frac{\partial b_{i,n}(x,s)}{\partial s}g_k'(s)ds.$

Using (5.20),we can deduce that $g_k(u_{i,n})$ is bounded in $W_0^{1,x}L_\varphi(Q_T)$ and $\frac{\partial B^{i,n}_{k,g}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T)$ independently of n.

thanks to (4.6) and properties of g_k, it follows that

$$|\int_{Q_T}\phi_{i,n}(x,t,u_n)g_k'(u_{i,n})dxdt|$$
$$\leq \|g_k'\|_\infty \int_{Q_T} c_i(x,t)\psi^{-1}\varphi(x,T_k(u_{i,n}))dxdt$$
$$\leq \|g_k'\|_\infty\Big(\psi^{-1}\varphi(x,k)\Big)\int_{Q_T} c_i(x,t)dxdt\Big) \leq C^1_{i,k}$$

By (5.13), we get

$$|\int_{Q_T} g_k''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}dxdt|$$
$$\leq \|g_k'\|_\infty\Big(C_{i,k} + c_0^i\int_{Q_T}\psi(x,\nabla T_k(u_{i,n}))dx\,dt\Big) \leq C^2_{i,k}$$

where $C^1_{i,k}$ and $C^2_{i,k}$ constants independently of n.

we conclude that $\frac{\partial g_k(u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T)$ for $k < n$. which implies that $g_k(u_{i,n})$ is compact in $L^1(Q_T)$.Due to the choice of g_k, we conclude that for each k,the sequence $T_k(u_{i,n})$ converges almost everywhere in Q_T,which implies that the sequence $u_{i,n}$ converge almost everywhere to some measurable function u_i in Q_T.

Then by the same argument in [9], we have

$$u_{i,n} \to u_i \text{ a.e. } Q_T, \tag{5.21}$$

where u_i is a measurable function defined on Q_T. and

$$b_{i,n}(x,u_{i,n}) \to b_i(x,u_i) \quad \text{a.e. in} \quad Q_T$$

by (5.15) and (5.21) we have

$$T_k(u_{i,n}) \to T_k(u_i) \tag{5.22}$$

weakly in $W_0^{1,x}L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ strongly in $E_\varphi(Q_T)$ and a.e in Q_T.

We now show that $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$. Indeed using $\frac{1}{\varepsilon}T_\varepsilon(u_{i,n})$ as a test function in (5.7),

$$\frac{1}{\varepsilon}\int_\Omega b^\varepsilon_{i,n}(x,u_{i,n})(t)\,dx + \frac{1}{\varepsilon}\int_{Q_T} a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt$$
$$-\frac{1}{\varepsilon}\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt + \frac{1}{\varepsilon}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})T_\varepsilon(u_{i,n})$$
$$= \frac{1}{\varepsilon}\int_\Omega b^\varepsilon_{i,n}(x,u_{i,0n})\,dx, \tag{5.23}$$

for almost any t in $(0,T)$. Where, $b^\varepsilon_{i,n}(r) = \int_0^r b'_{i,n}(s)T_\varepsilon(s)\,ds$. Since a_n satisfies (4.5) and $f_{i,n}$ satisfies (4.8), we get

$$\int_\Omega b^\varepsilon_{i,n}(x,u_{i,n})(t)\,dx \leq \int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt$$
$$+\int_\Omega b^\varepsilon_{i,n}(x,u_{i,0n})\,dx, \tag{5.24}$$

By Young inequality and (4.6), we get

$$\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt \leq \int_{|u_{i,n}|\leq\varepsilon}\psi(x,\Phi_{i,n}(x,t,u_{i,n}))\,dx\,dt$$
$$+\int_{|u_{i,n}|\leq\varepsilon}\varphi(x,\nabla T_\varepsilon(u_{i,n}))\,dx\,dt$$
$$\leq \varepsilon\psi(x,\frac{\alpha}{\lambda+1}\psi^{-1}\varphi(x,1)).meas(Q_T) + \int_{|u_{i,n}|\leq\varepsilon}(\varphi(x,\nabla T_\varepsilon(u_{i,n}))\,dx\,dt \tag{5.25}$$

Using the Lebesgue's Theorem and $\varphi(x,\nabla T_\varepsilon(u_{i,n})) \in W_0^{1,x}L(Q_T)$ in second term of the left hand side of the (5.25) and Letting $\varepsilon \to 0$ in (5.24)we obtain

$$\int_\Omega |b_{i,n}(x,u_{i,n})(t)|dx \le \|b_{i,n}(x,u_{i,0n})\|_{L^1(\Omega)} \quad (5.26)$$

for almost $t \in (0,T)$. thanks to (5.6) , (5.16), and passing to the limit-inf in (5.26), we obtain $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$. **Proof of** (5.18) **:**

Following the same way in([10]),we deduce that $a_n(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))$ is a bounded sequence in $(L_\psi(Q_T))^N$,and we obtain (5.18).

Proof of (5.19) **:**

Multiplying the approximating equation (5.7) by the test function $\theta_m(u_{i,n}) = T_{m+1}(u_{i,n}) - T_m(u_{i,n})$

$$\int_\Omega B_{i,m}(x,u_{i,n}(T))dx + \int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt$$
$$+\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt \quad (5.27)$$
$$+\int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt \le \int_\Omega B_{i,m}(x,u_{i,0n})dx,$$

where $B_{i,m}(x,r) = \displaystyle\int_0^r \theta_m(s)\frac{\partial b_{i,n}(x,s)}{\partial s}ds.$

By (4.6),we have

$$\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt$$
$$\le \int_{m\le|u_{i,n}|\le m+1} \psi(x,\frac{\beta}{\epsilon}\psi_x^{-1}\varphi(x,|u_{i,n}|))dx\,dt$$
$$+\epsilon\int_{m\le|u_{i,n}|\le m+1} \varphi(x,\nabla\theta_m(u_{i,n}))dx\,dt$$

Also $\int_{Q_T} f_{i,n}\theta_m(u_{i,n})dxdt \ge 0$ in view of (4.8).Then, The same argument in step 2 , we obtain,

$$\int_{Q_T} \varphi(x,\nabla u_{i,n})dxdt$$
$$\le C_i\Big(\int_{m\le|u_{i,n}|\le m+1} \psi(x,\frac{\beta}{\epsilon}\psi^{-1}\varphi(x,|u_{i,n}|))dx\,dt$$
$$+\int_\Omega B_{i,m}(x,u_{i,0n})dx\Big)$$

Where $C_i = \frac{1}{\alpha^i - \epsilon}$ where $0 < \epsilon < \alpha^i$.

passing to limit as $n \to +\infty$,since the pointwise convergence of $u_{i,n}$ and strongly convergence in $L^1(Q_T)$ of $B_{i,m}(x,u_{i,0n})$ we get

$$\lim_{n\to+\infty}\int_{Q_T} \varphi(x,\nabla u_{i,n})dxdt$$
$$\le C_i\Big(\int_{m\le|u_i|\le m+1} \psi(x,\frac{\beta}{\epsilon}\psi_x^{-1}\varphi(x,|u_i|))dx\,dt$$
$$+\int_\Omega B_{i,m}(x,u_{i,0})dx\Big)$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\le|u_i|\le m+1} \varphi(x,\nabla u_{i,n})dxdt = 0 \quad (5.28)$$

and the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dxdt$$
$$\le \lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\le|u_i|\le m+1} \varphi(x,\nabla\theta_m(u_{i,n}))dxdt$$
$$+\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\le|u_{i,n}|\le m+1} \psi(x,\phi_{i,n}(x,t,u_{i,n}))dxdt$$

Using the pointwise convergence of $u_{i,n}$ and by Lebesgue's theorem,in the second term of the right side ,we get

$$\lim_{n\to+\infty}\int_{m\le|u_{i,n}|\le m+1} \psi(x,\phi_{i,n}(x,t,u_{i,n}))dxdt$$
$$= \int_{m\le|u_i|\le m+1} \psi(x,\phi_i(x,t,u_i))dxdt,$$

and also ,by Lebesgue's theorem

$$\lim_{m\to+\infty}\int_{m\le|u_i|\le m+1} \psi(x,\phi_i(x,t,u_i))dxdt = 0 \quad (5.29)$$

we obtain with (5.28) and (5.29),

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})dxdt = 0$$

then passing to the limit in (5.27), we get the (5.19).

Step 3: Let $v_{i,j} \in \mathcal{D}(Q_T)$ be a sequence such that $v_{i,j} \to u_i$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence.

This specific time regularization of $T_k(v_{i,j})$ (for fixed $k \ge 0$) is defined as follows.

Let $(\alpha_{i,0}^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_{i,0}^\mu \in L^\infty(\Omega)\cap W_0^1L_\varphi(\Omega) \quad \text{for all}\mu > 0 \quad (5.30)$$
$$\|\alpha_{i,0}^\mu\|_{L^\infty(\Omega)} \le k \text{ for all}\mu > 0.$$

and $\alpha_{i,0}^\mu$ converges to $T_k(u_{i,0})$ a.e. in Ω and $\frac{1}{\mu}\|\alpha_{i,0}^\mu\|_{\varphi,\Omega}$ converges to 0 $\mu \to +\infty$.

For $k \ge 0$ and $\mu > 0$, let us consider the unique solution $(T_k(v_{i,j}))_\mu \in L^\infty(Q)\cap W_0^{1,x}L_\varphi(Q_T)$ of the monotone problem:

$$\frac{\partial(T_k(v_{i,j}))_\mu}{\partial t} + \mu((T_k(v_{i,j}))_\mu - T_k(v_{i,j})) = 0 \text{ in } D'(\Omega), \quad (5.31)$$
$$(T_k(v_{i,j}))_\mu(t=0) = \alpha_{i,0}^\mu \text{ in } \Omega. \quad (5.32)$$

Remark that due to

$$\frac{\partial(T_k(v_{i,j}))_\mu}{\partial t} \in W_0^{1,x}L_\varphi(Q_T) \quad (5.33)$$

We just recall that,

$$(T_k(v_{i,j}))_\mu \to T_k(u_i) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly}* \quad \text{in } L^\infty(Q_T), \tag{5.34}$$

$$(T_k(v_{i,j}))_\mu \to (T_k(u_i))_\mu \quad \text{in} \quad W_0^{1,x}L_\varphi(Q_T) \tag{5.35}$$

for the modular convergence as $j \to +\infty$.

$$(T_k(u_i))_\mu \to T_k(u_i) \quad \text{in} \quad W_0^{1,x}L_\varphi(Q_T) \tag{5.36}$$

for the modular convergence as $\mu \to +\infty$.

$$\|(T_k(v_{i,j}))_\mu\|_{L^\infty(Q_T)} \le max(\|(T_k(u_i))\|_{L^\infty(Q_T)}, \|\alpha_0^\mu\|_{L^\infty(\Omega)}) \le k \tag{5.37}$$

$\forall \ \mu > 0$,$\forall \ k > 0$. Now, we introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that, for any $m \ge 1$

$$S_m(r) = r \text{ for } |r| \le m, \quad \text{supp}(S'_m) \subset [-(m+1), (m+1)], \tag{5.38}$$

$$\|S''_m\|_{L^\infty(\mathbb{R})} \le 1.$$

Through setting, for fixed $K \ge 0$,

$$W^n_{i,j,\mu} = T_K(u_{i,n}) - T_K(v_{i,j})_\mu \quad \text{and} \quad W^n_{i,\mu} = T_K(u_{i,n}) - T_K(u_i)_\mu \tag{5.39}$$

we obtain upon integration,

$$\begin{aligned}
&\int_{Q_T} \Big\langle \frac{\partial b_{i,S_m}(u_{i,n})}{\partial t}, W^n_{i,j,\mu} \Big\rangle dx\,dt \\
&+ \int_{Q_T} S'_m(u_{i,n}) a_n(x, u_i^n, \nabla u_{i,n}) \nabla W^n_{i,j,\mu}\,dx\,dt \\
&+ \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x, u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n}\,dx\,dt \\
&+ \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n}) S'_m(u_{i,n}) \nabla W^n_{i,j,\mu}\,dx\,dt \\
&+ \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} \Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n}\,dx\,dt \\
&+ \int_{Q_T} f_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu}\,dx\,dt = 0
\end{aligned} \tag{5.40}$$

Next we pass to the limit as n tends to $+\infty$, j tends to $+\infty$, μ tends to $+\infty$ and then m tends to $+\infty$, the real number $K \ge 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \ge 0$:

$$\liminf_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} \Big\langle \frac{\partial b_{i,S_m}(u_{i,n})}{\partial t}, S'_m(u_{i,n}) W^n_{i,j,\mu} \Big\rangle \ge 0, \tag{5.41}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} S'_n(u_{i,n}) \Phi_{i,n}(x,t,u_{i,n}) \nabla W^n_{i,j,\mu} = 0, \tag{5.42}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} S''_m(u_{i,n}) W^n_{i,\mu} \Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n} = 0, \tag{5.43}$$

$$\lim_{m\to+\infty} \overline{\lim_{\mu\to+\infty}} \lim_{j\to+\infty} \overline{\lim_{n\to+\infty}} \Big| \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x,t,u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n} \Big| = 0 \tag{5.44}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} f_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu} = 0. \tag{5.45}$$

$$\limsup_{n\to+\infty} \int_{Q_T} a(x,t,u_{i,n}, \nabla T_K(u_{i,n})) \nabla T_K(u_{i,n})\,dx\,dt \tag{5.46}$$

$$\le \int_{Q_T} X_{i,K} \nabla T_K(u_i)\,dx\,dt. \tag{5.47}$$

$$\int_{Q_T} [a(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})) - a(x,t,T_k(u_{i,n}), \nabla T_k(u_i))]$$

$$[\nabla T_k(u_{i,n}) - \nabla T_k(u_i)]\,dx\,dt \to 0. \tag{5.48}$$

Proof of (5.41):

Lemma 5.1

$$\int_{Q_T} \Big\langle \frac{\partial b_{i,n}(x, u_{i,n})}{\partial t}, S'_m(u_{i,n}) W^n_{i,j,\mu} \Big\rangle dx\,dt \ge \epsilon(n, j, \mu, m), \tag{5.49}$$

See [23]. **Proof of** (5.42):
If we take $n > m+1$, we get

$$\phi_{i,n}(x,t,u_{i,n}) S'_m(u_{i,n}) = \phi_i(x,t,T_{m+1}(u_{i,n})) S'_m(u_{i,n})$$

Using (4.6), we have:

$$\psi(\phi_{i,n}(x,t,T_{m+1}(u_{i,n}) S'_m(u_{i,n})) \le (m+1) \psi(\phi_i(x,t,T_{m+1}(u_{i,n})))$$

$$\le (m+1) \psi(\|c(x,t)\|_{L^\infty(Q_T)} \psi^{-1} M(m+1))$$

Then $\phi_{i,n}(x,t,u_n) S_m(u_{i,n})$ is bounded in $L_\psi(Q_T)$, thus, by using the pointwise convergence of $u_{i,n}$ and Lebesgue's theorem we obtain $\phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \to \phi_i(x,t,u_i) S_m(u_i)$ with the modular convergence as $n \to +\infty$, then $\phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \to \phi(x,t,u_i) S_m(u_i)$ for $\sigma(\prod L_\psi, \prod L_\varphi)$.
In the other hand $\nabla W^n_{i,j,\mu} = \nabla T_k(u_{i,n}) - \nabla(T_k(v_{i,j}))_\mu$ for converge to $\nabla T_k(u_i) - \nabla(T_k(v_{i,j}))_\mu$ weakly in $(L_\varphi(Q_T))^N$,then

$$\int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \nabla W^n_{i,j,\mu}\,dx\,dt$$

$$\to \int_{Q_T} \phi_i(x,t,u_i) S_m(u_i) \nabla W_{i,j,\mu}\,dx\,dt$$

as $n \to +\infty$.
By using the modular convergence of $W_{i,j,\mu}$ as $j \to +\infty$ and letting μ tends to infinity, we get (5.42).
Proof of (5.43):
For $n > m+1 > k$, we have $\nabla u_{i,n} S''_m(u_{i,n}) = \nabla T_{m+1}(u_{i,n})$ a.e. in Q_T. By the almost every where convergence of $u_{i,n}$ we have $W^n_{i,j,\mu} \to W_{i,j,\mu}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\phi_{i,n}(x,t,T_{m+1}(u_{i,n})))_n$ converge strongly in $E_\psi(Q_T)$ then

$$\phi_{i,n}(x,t,T_{m+1}(u_{i,n}))\ W^n_{i,j,\mu} \to \phi_i(x,t,T_{m+1}(u_i))\ W_{i,j,\mu}$$

converge strongly in $E_\psi(Q_T)$ as $n \to +\infty$.By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u_i)$ weakly in $(L_\varphi(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{m\le|u_{i,n}|\le m+1} \phi_{i,n}(x,t,T_{m+1}(u_{i,n}))\nabla u_{i,n} S''_m(u_{i,n}) W^n_{i,j,\mu}\,dx\,dt$$

$$\to \int_{m\le|u_i|\le m+1} \phi(x,t,u_i))\nabla u_i W_{i,j,\mu}\,dx\,dt$$

as $n \to +\infty$
with the modular convergence of $W_{i,j,\mu}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (5.43).
Proof of (5.44):
For any $m \ge 1$ fixed, we have

$$\Big|\int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} W^n_{i,j,\mu}\,dx\,dt\Big|$$

$$\le \|S''_m\|_{L^\infty(\mathbb{R})}\|W^n_{i,j,\mu}\|_{L^\infty(Q_T)} \int_{\{m\le|u_{i,n}|\le m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n}) \times \nabla u_{i,n}\,dx\,dt,$$

for any $m \ge 1$, and any $\mu > 0$. In view (5.37) and (5.38), we can obtain

$$\limsup_{n\to+\infty}\Big|\int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} W^n_{i,j,\mu}\,dx\,dt\Big| \le 2K\limsup_{n\to+\infty}\int_{\{m\le|u_{i,n}|\le m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt, \tag{5.50}$$

for any $m \ge 1$. Using (5.19) we pass to the limit as $m \to +\infty$ in (5.50) and we obtain (5.44).
Proof of (5.45):
For fixed $n \ge 1$ and $n > m+1$, we have

$$f_{1,n}(x,u_{1,n},u_{2,n})S'_m(u_{1,n}) = f_1(x,T_{m+1}(u_{1,n}),T_n(u_{2,n}))S'_m(u_{1,n}),$$

$$f_{2,n}(x,u_{1,n},u_{2,n})S'_m(u_{2,n}) = f_2(x,T_n(u_{1,n}),T_{m+1}(u_{2,n}))S'_m(u_{2,n}),$$

In view (4.9),(4.10),(5.22) and Lebesgue's the theorem allow us to get, for

$$\lim_{n\to+\infty}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt$$

Using (5.35), we follow a similar way we get as $j \to +\infty$,

$$\lim_{j\to+\infty}\int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt$$

we fixed $m > 1$, and using (5.36), we have

$$\lim_{\mu\to+\infty}\int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt = 0$$

Then we conclude the proof of (5.45).
Proof of (5.46):
If we pass to the lim-sup when n ,j and μ tends to $+\infty$ and then to the limit as m tends to $+\infty$ in (5.40). We obtain using (5.41)-(5.45), for any $K \ge 0$,

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) \Big(\nabla T_K(u_{i,n}) - \nabla T_K(v_{i,j})_\mu\Big)\,dx\,dt \le 0.$$

Since

$$S'_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n}) = a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n})$$

for $n > K$ and $K \le m$. Then, for $K \le m$,

$$\limsup_{n\to+\infty}\int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n})\,dx\,dt \le \lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n}) a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_K(v_{i,j})_\mu\,dx\,dt. \tag{5.51}$$

Thanks to (5.38), we have in The right hand side of (5.51), for $n > m+1$,

$$S'_m(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'_m(u_{i,n})a\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big) \text{ a.e. in } Q_T.$$

Using (5.18), and fixing $m \ge 1$, we get

$$S'_m(u_{i,n})a_n(u_{i,n},\nabla u_{i,n}) \rightharpoonup S'_m(u_i)X_{i,m+1} \text{ weakly in } (L_\psi(Q_T))^N.$$

when $n \to +\infty$.
We pass to limit as $j \to +\infty$ and $\mu \to +\infty$, and using (5.35)-(5.36)

$$\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n}))a_n(x,t,u_{i,n},\nabla u_{i,n}))\nabla T_K(v_{i,j})_\mu\,dx\,dt = \int_{Q_T} S'_m(u_i)X_{i,m+1}\nabla T_K(u_i)\,dx\,dt = \int_{Q_T} X_{i,m+1}\nabla T_K(u_i)\,dx\,dt \tag{5.52}$$

where $K \leq m$, since $S'_m(r) = 1$ for $|r| \leq m$.
On the other hand, for $K \leq m$, we have

$$a\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big)\chi_{\{|u_{i,n}|<K\}}$$

$$= a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\chi_{\{|u_{i,n}|<K\}},$$

a.e. in Q_T. Passing to the limit as $n \to +\infty$, we obtain

$$X_{i,m+1}\chi_{\{|u_i|<K\}} = X_{i,K}\chi_{\{|u_i|<K\}} \quad \text{a.e. in } Q_T - \{|u_i| = K\} \text{ for } K \leq n. \tag{5.53}$$

Then

$$X_{m+1}\nabla T_K(u_i) = X_K\nabla T_K(u_i) \quad \text{a.e. in } Q_T. \tag{5.54}$$

Then we obtain (5.46).
Proof of (5.48):
Let $K \geq 0$ be fixed. Using (4.5) we have

$$\int_{Q_T}\Big[a(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u_i))\Big]$$
$$\Big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\Big]dx\,dt \geq 0, \tag{5.55}$$

In view (1.1) and (5.22), we get

$$a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{a.e. in } Q_T,$$

as $n \to +\infty$, and by (4.2) and Lebesgue's theorem, we obtain

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big) \to a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big) \tag{5.56}$$

strongly in $(L_\psi(Q_T))^N$. Using (5.46), (5.22), (5.18) and (5.56), we can pass to the lim-sup as $n \to +\infty$ in (5.55) to obtain (5.48).
To finish this step, we prove this Lemma:

Lemma 5.2 *For $i = 1,2$ and fixed $K \geq 0$, we have*

$$X_{i,K} = a\Big(xt,,T_K(u_i),\nabla T_K(u_i)\Big) \quad \text{a.e. in } Q. \tag{5.57}$$

Also, as $n \to +\infty$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n}) \rightharpoonup a\Big(x,t,T_K(u_i),DT_K(u_i)\Big)\nabla T_K(u_i), \tag{5.58}$$

weakly in $L^1(Q_T)$.

Proof of (5.57):
It's easy to see that

$$a_n(x,t,T_K(u_{i,n}),\xi) = a(x,t,T_K(u_{i,n}),\xi) = a_K(x,t,T_K(u_{i,n}),\xi)$$

a.e. in Q_T
for any $K \geq 0$, any $n > K$ and any $\xi \in \mathbb{R}^N$.
In view of (5.18), (5.48) and (5.56) we obtain

$$\lim_{n\to+\infty}\int_{Q_T} a_K\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})\,dx\,dt$$
$$= \int_{Q_T} X_{i,K}\nabla T_K(u_i)\,dx\,dt. \tag{5.59}$$

Since (1.1), (4.4) and (5.22), imply that the function $a_K(x,s,\xi)$ is continuous and bounded with respect to s. Then we conclude that (5.57).
Proof of (5.58):
Using (4.5) and (5.48), for any $K \geq 0$ and any $T' < T$, we have

$$\Big[a(x,t,T_K(u_{i,n},\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u))\Big]$$
$$\times\Big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\Big] \to 0 \tag{5.60}$$

strongly in $L^1(Q_{T'})$ as $n \to +\infty$.
On the other hand with (5.22), (5.18), (5.56) and (5.57), we get

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_i)$$
$$\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_{i,n})$$
$$\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$
$$\to a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i),$$

strongly in $L^1(Q)$, as $n \to +\infty$.
It's results from (5.60), for any $K \geq 0$ and any $T' < T$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})$$
$$\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i) \tag{5.61}$$

weakly in $L^1(Q_{T'})$ as $n \to +\infty$.then for $T' = T$, we have (5.58). Finally we should prove that u_i satisfies (4.13).
Step 4:Pass to the limit.
we first show that u satisfies (4.13)

$$\int_{m\leq|u_{i,n}|\leq m+1\}} a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt$$
$$= \int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\Big[\nabla T_{m+1}(u_{i,n}) - \nabla T_m(u_{i,n})\Big]dx\,dt$$
$$= \int_{Q_T} a_n\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big)\nabla T_{m+1}(u_{i,n})\,dx\,dt$$
$$- \int_{Q_T} a_n\Big(x,t,T_m(u_{i,n}),\nabla T_m(u_{i,n})\Big)\nabla T_m(u_{i,n})\,dx\,dt$$

for $n > m+1$. According to (5.58), one can pass to the limit as $n \to +\infty$; for fixed $m \geq 0$ to obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt \\
&= \int_Q a\Big(x,t,T_{m+1}(u_i),\nabla T_{m+1}(u_i)\Big)\nabla T_{m+1}(u_i)\,dx\,dt \\
&\quad - \int_Q a\Big(x,t,T_m(u_i),\nabla T_m(u_i)\Big)\nabla T_m(u_i)\,dx\,dt \\
&= \int_{m\leq|u_i|\leq m+1\}} a(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt
\end{aligned} \tag{5.62}$$

Pass to limit as m tends to $+\infty$ in (5.62) and using (5.19) show that u_i satisfies (4.13).

Now we shown that u_i to satisfy (4.14)and (4.15). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\operatorname{supp} S' \subset [-K,K]$. the Pointwise multiplication of the approximate equation (1.1) by $S'(u_{i,n})$ leads to

$$\begin{aligned}
&\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t} - \operatorname{div}\Big(S'(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\Big) \\
&+ S''(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \\
&- \operatorname{div}\Big(S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\Big) \\
&+ S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \\
&= f_{i,n}(x,u_{1,n},u_{1,n})S'(u_{i,n})
\end{aligned} \tag{5.63}$$

in $D'(Q_T)$, for $i = 1,2$.

Now we pass to the limit in each term of (5.63).

Limit of $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$: Since $B^n_{i,S}(u_{i,n})$ converges to $B_{i,S}(u_i)$ a.e. in Q_T and in $L^\infty(Q_T)$ weak $\star$ and S is bounded and continuous. Then $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$ converges to $\frac{\partial b_{i,S}(u_i)}{\partial t}$ in $D'(Q_T)$ as n tends to $+\infty$.

Limit of $div\Big(S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\Big)$: Since $\operatorname{supp} S' \subset [-K,K]$, for $n > K$, we have

$$S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})$$

$$= S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)$$

a.e. in Q_T. Using the pointwise convergence of $u_{i,n}$, (5.38),(5.18) and (5.57), imply that

$$S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)$$

$$\rightharpoonup S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)$$

weakly in $(L_\psi(Q_T))^N$, for $\sigma(\Pi L_\psi,\Pi E_\varphi)$ as $n \to +\infty$, since $S'(u_i) = 0$ for $|u_i| \geq K$ a.e. in Q_T. And

$$S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big) = S'(u_i)a(x,t,u_i,\nabla u_i)$$

$$\text{a.e. in}\quad Q_T.$$

Limit of $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$. Since $\operatorname{supp} S'' \subset [-K,K]$, for $n > K$, we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$$

$$= S''(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n}) \quad \text{a.e. in } Q_T.$$

The pointwise convergence of $S''(u_{i,n})$ to $S''(u_i)$ as $n \to +\infty$, (5.38) and (5.58) we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$$

$$\rightharpoonup S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$, as $n \to +\infty$, and

$$S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

$$= S''(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i \quad \text{a.e.in } Q_T.$$

Limit of $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})$: We have

$$S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})$$

$$= S'(u_{i,n})\Phi_{i,n}(x,t,T_K(u_{i,n}))$$

a.e.in Q_T, Since $\operatorname{supp} S' \subset [-K,K]$.Using (4.5), (5.24) and (5.16), it's easy to see that $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \rightharpoonup S'(u_i)\Phi_i(x,t,T_K(u_i))$ weakly for $\sigma(\Pi L_\psi,\Pi L_\varphi)$ as $n \to +\infty$. And $S'(u_i)\Phi_i(x,t,T_K(u_i)) = S'(u_i)\Phi_i(x,t,u_i)$ a.e. in Q_T.

Limit of $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}$: Since $S' \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} S' \subset [-K,K]$, we have $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = \Phi_{i,n}(x,t,T_K(u_{i,n}))\nabla S'(T_K(u_{i,n}))$ a.e. in Q_T. The weakly convergence of truncation allows us to prove that

$$S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \rightharpoonup \Phi_i(x,t,u_i)\nabla S'(u_i),$$

strongly in $L^1(Q_T)$.

Limit of $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n})$: Using (4.9), (4.10), (5.4) and (5.5), we have $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n}) \to f_i(x,u_1,u_2)S'(u_i)$ strongly in $L^1(Q_T)$, as $n \to +\infty$.

It remains to show that for i=1,2 $B_S(x,u_i)$ satisfies the initial condition (4.15).

To this end, firstly remark that, in view of the definition of S'_φ, we have $B_\varphi(x,u_{i,n})$ is bounded in $L^\infty(Q_T)$.

Secondly, by (5.41) we show that $\dfrac{\partial B_\varphi(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T))$. As a consequence, an Aubin's type Lemma (see e.g., [11], Corollary 4) implies that $B_\varphi(x,u_{i,n})$ lies in a compact set of $C^0([0,T];L^1(\Omega))$.

It follows that, on one hand,$B_\varphi(x,u_i,n)(t=0)$ converges to $B_\varphi(x,u_i)(t=0)$ strongly in $L^1(\Omega)$. On the order hand, the smoothness of B_φ imply that $B_\varphi(x,u_{i,n})(t=0)$ converges to $B_\varphi(x,u_i)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $B_\varphi(x,u_{i,n})(t=0) = B_\varphi(x,u_{i,0n})$ converges to $B_\varphi(x,u_i)(t=0)$ strongly in $L^1(\Omega)$, we obtain $B_\varphi(x,u_i)(t=0) = B_\varphi(x,u_{i,0})$ a.e. in Ω and for all $M > 0$, now letting M to $+\infty$, we conclude that $b(x,u_i)(t=0) = b(x,u_{i,0})$ a.e. in Ω.

As a conclusion, the proof of Theorem (5.1) is complete.

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