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Existence Results for Nonlinear Anisotropic Elliptic Equation

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ABSTRACT

In this work, we shall be concerned with the existence of weak solutions of anisotropic elliptic operators $Au+\sum_{i=1}^{N} g_i(x,u,\nabla u)+\sum_{i=1}^{N} H_i(x,\nabla u) = f-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}k_i$, *where the right hand side* f *belongs to* $L^{p'_\infty}(\Omega)$ *and* k_i *belongs to* $L^{p'_i}(\Omega)$ *for* $i=1,...,N$ *and* A *is a Leray-Lions operator. The critical growth condition on* g_i *is the respect to* ∇u *and no growth condition with respect to* u, *while the function* H_i *grows as* $|\nabla u|^{p_i-1}$.

1 Introduction

In this paper we study the existence of weak solutions to anisotropic elliptic equations with homogeneous Dirichlet boundary conditions of the type

$$\begin{cases} Au+\sum_{i=1}^{N} g_i(x,u,\nabla u)+\sum_{i=1}^{N} H_i(x,\nabla u) \\ \qquad\qquad = f-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}k_i & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with Lipschitz continuous boundary. The operator $Au = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)$ is a Leray-Lions operator such that the functions a_i, g_i and H_i are the Carathodory functions satisfying the following conditions for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N, \xi'\in\mathbb{R}^N$ and a.e. in Ω :

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \lambda\sum_{i=1}^{N}|\xi_i|^{p_i},$$

$$|a_i(x,s,\xi)|\leq\gamma[|s|^{\frac{p_\infty}{p'_i}}+|\xi_i|^{p_i-1}],$$

$$(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi'_i)>0 \quad \text{for } \xi_i\neq\xi'_i,$$

$$g_i(x,s,\xi)s\geq 0,$$

$$|g_i(x,s,\xi)|\leq L(|s|)|\xi_i|^{p_i}\ \forall i=1,....,N,$$

$$|H_i(x,\xi)|\leq b_i|\xi_i|^{p_i-1},$$

where λ,γ,b_i are some positive constants, for $i=1,...,N$ and $L:\mathbb{R}^+\to\mathbb{R}^+$is a continuous and non decreasing function. The right hand side f and k_i for $i=1,...,N$ are functions belonging to $L^{p'_\infty}(\Omega)$ and $L^{p'_i}(\Omega)$ where $p'_i=\frac{p_i}{p_i-1}, p'_\infty=\frac{p_\infty}{p_\infty-1}$ with $p_\infty=\max\{\overline{p}*,p_+\}$ where $p_+=\max\{p_1,...,p_N\}$, $\overline{p}=\frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}}$ and $\overline{p}*=\frac{N\overline{p}}{N-\overline{p}}$.

Since the growth and the coercivity conditions of each a_i for all $i=1,...,N$ depend on p_i, we have need to use the anisotropic Sobolev space. We mention some papers on anisotropic Sobolev spaces (see e.g.[1]-[5]).

If $p_i=p$ for all $i=1,...,N$, we refer some works such as by Guibé in [6], by Monetti and Randazzo in [7] and by Y. Akdim, A. Benkirane and M. El Moumni in [8].

In [3], L.Boccardo, T. Gallouet and P. Marcellini have studied the problem (1) when $a_i(x,u,\nabla u) = \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\frac{\partial u}{\partial x_i}$, $g_i=0$, $H_i=0$, $k_i=0$ and $f=\mu$ is Radon's measure. In [5], F. Li has proved the existence and regularity of weak solutions of the problem (1) with $g_i=0$, $H_i=0$, $k_i=0$ for all $i=1,...,N$ and f belongs to $L^m(\Omega)$ with $m>1$. In [9], R. Di Nardo and F. Feo have proved the existence of weak solution of the problem (1) when $g_i=0$ for all $i=1,...,N$. In [10], we have proved the existence and uniqueness of weak solution of the problem (1) but when $Au=-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\nabla u)$ (a_i depending only on x and ∇u).

In this work, we prove the existence of weak solu-

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tions of the problem (1), based on techniques related to that of Di castro in [11] and to the recent work's Di Nardo and F. Feo in [9].

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary and let $1 < p_1, ..., p_N < \infty$ be N real numbers, $p^+ = \max\{p_1, ..., p_N\}, p^- = \min\{p_1, ..., p_N\}$ and $\overrightarrow{p} = (p_1, ..., p_N)$. The anisotropic Sobolev space (see [12])

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, 2, ..., N\right\}$$

is a Banach space with respect to norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}.$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. We recall a Poincarr-type inequality. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ then there exists a constant C_p such that (see[13])

$$\|u\|_{L^{p_i}(\Omega)} \leq C_p \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)} \text{ for } i = 1, ..., N. \quad (1)$$

Moreover a Sobolev-type inequality holds. Let us denote by $\overline{p}$ the harmonic mean of these numbers, i.e. $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i}$. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then there exists (see [12]) a constant C_s such that

$$\|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}. \quad (2)$$

Where $q = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p} < N$ or $q \in [1, +\infty[$ if $\overline{p} \geq N$. We recall the arithmetic mean: Let $a_1, ..., a_N$ be positive numbers, it holds

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N} a_i. \quad (3)$$

Which implies by (2)

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N}\sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}. \quad (4)$$

When $\overline{p} < N$ hold, inequality (4) implies the continuous embedding of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$. On the other hand the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ relies on inequality (1). Let us put $p_\infty := \max\{\overline{p}*, p^+\}$

Proposition 1 *For $q \in [1, p_\infty]$ there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$. If $q < p_\infty$ the embedding is compact.*

3 Assumptions and Definition

We consider the following class of nonlinear anisotropic elliptic homogenous Dirichlet problems

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + \sum_{i=1}^{N} g_i(x, u, \nabla u) + \\ \sum_{i=1}^{N} H_i(x, \nabla u) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i \text{ in } \Omega, \\ \quad u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary $\partial\Omega$, $1 < p_1, ..., p_N < \infty$. We assume that a_i: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, g_i: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and H_i: $\Omega \times \mathbb{R}^N \to \mathbb{R}$ are Carathodory functions such that for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N$ and a. e. in Ω :

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i}, \quad (5)$$

$$|a_i(x, s, \xi)| \leq \gamma[|s|^{\frac{p_\infty}{p_i'}} + |\xi_i|^{p_i - 1}], \quad (6)$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for } \xi_i \neq \xi_i', \quad (7)$$

$$g_i(x, s, \xi)s \geq 0, \quad (8)$$

$$|g_i(x, s, \xi)| \leq L(|s|)|\xi_i|^{p_i} \; \forall i = 1, ..., N, \quad (9)$$

$$|H_i(x, \xi)| \leq b_i|\xi_i|^{p_i - 1}, \quad (10)$$

where λ, γ, b_i are some positive constants, for $i = 1, ..., N$ and $L : \mathbb{R}^+ \to \mathbb{R}^+$is a continuous and non decreasing function. Moreover, we suppose that

$$f \in L^{p_\infty'}(\Omega), \quad (11)$$

$$k_i \in L^{p_i'}(\Omega) \text{ for } i = 1, ..., N. \quad (12)$$

Definition 1 *A function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a weak solution of the problem (1) if $\sum_{i=1}^{N} g_i(x, u, \nabla u) \in L^1(\Omega)$ and u satisfies*

$$\sum_{i=1}^{N} \int_\Omega \left[a_i(x, u, \nabla u)\frac{\partial\varphi}{\partial x_i} + g_i(x, u, \nabla u)\varphi + H_i(x, \nabla u)\varphi\right]$$

$$= \int_\Omega \left[f\varphi + \sum_{i=1}^{N} k_i \frac{\partial\varphi}{\partial x_i}\right]$$

$\forall \varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.

4 Main results

In this section we prove the existence of at least a weak solution of the problem (1). We consider the approximate problems.

4.1 Approximate problems and a priori estimates

Let

$$g_i^n(x,u,\nabla u)=\frac{g_i(x,u,\nabla u)}{1+\frac{1}{n}|g_i(x,u,\nabla u)|}$$

and

$$H_i^n(x,\nabla u)=\frac{H_i(x,\nabla u)}{1+\frac{1}{n}|H_i(x,\nabla u)|}.$$

By Leray-Lions (see e.g. [14]), there exists at least a weak solution $u_n\in W_0^{1,\overrightarrow{p}}(\Omega)$ of the following approximate problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}a_i(x,u_n,\nabla u_n)+\sum_{i=1}^N g_i^n(x,u_n,\nabla u_n) \\ \quad +\sum_{i=1}^N H_i^n(x,\nabla u_n)=f-\sum_{i=1}^N\frac{\partial}{\partial x_i}k_i \quad \text{in } \Omega \\ u_n=0 \quad \text{on } \partial\Omega. \end{cases} \tag{13}$$

Lemma 1 *See ([9], lemma 4.2) Let $A\in\mathbb{R}^+$ and $u\in W_0^{1,\overrightarrow{p}}(\Omega)$, then there exists t measurable subsets $\Omega_1,...,\Omega_t$ of Ω and t functions $u_1,...,u_t$ such that $\Omega_i\cap\Omega_j=\emptyset$ for $i\neq j$, $|\Omega_t|\leq A$ and $|\Omega_s|=A$ for $s\in\{1,...,t-1\}$, $\{x\in\Omega: |\frac{\partial u_s}{\partial x_i}|\neq 0$ for $i=1,...,N\}\subset\Omega_s$, $\frac{\partial u}{\partial x_i}=\frac{\partial u_s}{\partial x_i}$ a. e. in Ω_s, $\frac{\partial(u_1+...+u_s)}{\partial x_i}u_s=(\frac{\partial u}{\partial x_i})u_s$, $u_1+...+u_s=u$ in Ω and $sign(u)=sign(u_s)$ if $u_s\neq 0$ for $s\in\{1,...,t\}$ and $i\in\{1,...,N\}$.*

Proposition 2 *Assume that $\overline{p}<N$, (5)-(12) hold and let $u_n\in W_0^{1,\overrightarrow{p}}(\Omega)$ be a solution to problem (13) then, we have*

$$\sum_{i=1}^N\int_\Omega|\frac{\partial u_n}{\partial x_i}|^{p_i}\leq C, \tag{14}$$

for some positive constant C depending on N, Ω, λ, γ, p_i, b_i, $\|f\|_{L^{p'_\infty}(\Omega)}$, $\|g_i\|_{L^{p'_i}(\Omega)}$ for $i=1,...,N$.

Proof: Let A be a positive real number, that will be chosen later, Referring to lemma 1. Let us fix $s\in\{1,...,t\}$ and let us use $T_k(u_s)$ as test function in problem 13, using (5), (8), Young's and Hölder's inequalities and proposition 1 we obtain

$$\sum_{i=1}^N\int_{\{u_s\leq k\}}|\frac{\partial u_s}{\partial x_i}|^{p_i} \tag{15}$$

$$\leq C_1\Big(\|f\|_{L^{p'_\infty}(\Omega)}d_s^{\frac{1}{N}}+\sum_{i=1}^N\int_\Omega|H_i(x,\nabla u)||u_s|+\sum_{i=1}^N\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}\Big).$$

The dominated convergence theorem implies that

$$\sum_{i=1}^N\int_\Omega|\frac{\partial u_s}{\partial x_i}|^{p_i}\leq C_1\Big(\|f\|_{L^{p'_\infty}(\Omega)}d_s^{\frac{1}{N}}+$$

$$\sum_{i=1}^N\int_\Omega|H_i(x,\nabla u)||u_s|+\sum_{i=1}^N\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}\Big),$$

for some constant $C_1>0$, where $d_s=\prod_{i=1}^N\Big(\int_\Omega|\frac{\partial u_s}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}$ here and in what follows the constants depend on the data but not on the function u.

Using condition (10), Hölder's and Young's inequalities, lemma 1 and proposition 1 we get

$$\sum_{i=1}^N\int_\Omega|H_i(x,\nabla u)||u_s|\leq\sum_{i=1}^N\int_\Omega b_i|\frac{\partial u}{\partial x_i}|^{p_i-1}|u_s|$$

$$\leq C_2\sum_{i=1}^N\Big[\sum_{\sigma=1}^s\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i-1}|u_s| +\int_{\Omega\setminus\cup_{\sigma=1}^s\Omega_\sigma}|\frac{\partial u}{\partial x_i}|^{p_i-1}|u_s|\Big]$$

$$\leq C_2\sum_{i=1}^N\Big[\sum_{\sigma=1}^s\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i-1}|u_s|\Big]$$

$$\leq C_2\sum_{i=1}^N\sum_{\sigma=1}^s\Big(\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p'_i}}\Big(\int_\Omega|u_s|^{p_\infty}\Big)^{\frac{1}{p_\infty}}|\Omega_\sigma|^{\frac{1}{p_i}-\frac{1}{p_\infty}}$$

$$\leq C_2\sum_{i=1}^N\sum_{\sigma=1}^s\Big[\frac{1}{p'_i}\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i} +\frac{1}{p_i}\Big(\int_\Omega|u_s|^{p_\infty}\Big)^{\frac{p_i}{p_\infty}}\Big]|\Omega_\sigma|^{\frac{1}{p_i}-\frac{1}{p_\infty}}$$

since $\Big(\|u_s\|_{L^{\overline{p}^*}(\Omega)}\leq C_s\prod_{j=1}^N\|\frac{\partial u_s}{\partial x_j}\|^{\frac{1}{N}}_{L^{p_j}(\Omega)}$ and $|\Omega_\sigma|\leq A\Big)$,

hence

$$\sum_{i=1}^N\int_\Omega|H_i(x,\nabla u)||u_s| \tag{16}$$

$$\leq C_2\sum_{i=1}^N A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\sum_{\sigma=1}^s\Big[\frac{1}{p'_i}\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i} +\frac{CC_s}{p_i}\prod_{j=1}^N\|\frac{\partial u_s}{\partial x_j}\|^{\frac{p_i}{N}}_{L^{p_j}(\Omega)}\Big]$$

$$\leq C_3\sum_{i=1}^N A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\Big[\int_{\Omega_s}|\frac{\partial u_s}{\partial x_i}|^{p_i}+\sum_{\sigma=1}^{s-1}\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i}+d_s^{\frac{p_i}{N}}\Big]$$

for some constant $C_3>0$. Putting (16) in (15) we obtain

$$\sum_{i=1}^N\int_\Omega|\frac{\partial u_s}{\partial x_i}|^{p_i}\leq C_1\Big\{\|f\|_{L^{p'_\infty}(\Omega)}d_s^{\frac{1}{N}}+\sum_{i=1}^N\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}+$$

$$C_3\sum_{i=1}^N A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\Big[\int_{\Omega_s}|\frac{\partial u_s}{\partial x_i}|^{p_i}+\sum_{\sigma=1}^{s-1}\int_{\Omega_\sigma}|\frac{\partial u_\sigma}{\partial x_i}|^{p_i}+d_s^{\frac{p_i}{N}}\Big]\Big\}. \tag{17}$$

If A is such that

$$1-C_1C_3\sum_{i=1}^N A^{\frac{1}{p_i}-\frac{1}{p_\infty}}>0, \tag{18}$$

inequality (17) becomes

$$\sum_{i=1}^{N}\int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i} \le C_4\Big\{\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} + \sum_{\sigma=1}^{s-1}\sum_{i=1}^{N} A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\Big(\sum_{j=1}^{N}\int_{\Omega_s}|\frac{\partial u_s}{\partial x_j}|^{p_j}\Big) + \sum_{i=1}^{N} A^{\frac{1}{p_i}-\frac{1}{p_\infty}} d_s^{\frac{p_i}{N}}\Big\} \quad (19)$$

for some constant $C_4 > 0$ and for $s = 1$, we get

$$\int_\Omega |\frac{\partial u_1}{\partial x_i}|^{p_i} \le \sum_{i=1}^{N}\int_\Omega |\frac{\partial u_1}{\partial x_i}|^{p_i}$$

$$\le C_4\Big\{\|f\|_{L^{p'_\infty}(\Omega)} d_1^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} + \sum_{i=1}^{N} A^{\frac{1}{p_i}-\frac{1}{p_\infty}} d_1^{\frac{p_i}{N}}\Big\}. \quad (20)$$

Let us choose A such that (18) and $1 - C_4\sum_{i=1}^{N} A^{\frac{N}{p_i}(\frac{1}{p_i}-\frac{1}{p_\infty})} > 0$ hold, (see [9]).
For example, we can take

$$A < \min\Big\{1, (\tfrac{1}{2C_3})^{\frac{1}{\min_{i=1...N}\{\frac{1}{p_i}-\frac{1}{p_\infty}\}}} ; (\tfrac{1}{2C_3})^{\frac{1}{\min_{i=1...N}\{\frac{N}{p_i}(\frac{1}{p_i}-\frac{1}{p_\infty})\}}}\Big\}.$$

By this choice, we obtain

$$d_1 = \prod_{i=1}^{N}\Big(\int_\Omega |\frac{\partial u_1}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}$$

$$\le C_5\Big[\Big(\|f\|^{\frac{N}{\overline{p}}}_{L^{p'_\infty}(\Omega)} + \|\nu_2\|^{\frac{N}{\overline{p}}}_{L^{p'_\infty}(\Omega)}\Big) d_1^{\frac{1}{\overline{p}}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}\Big].$$ Then there exists a constant $C_6 > 0$ such that $d_1 \le C_6$ and by (20), we obtain

$$\sum_{i=1}^{N}\int_\Omega |\frac{\partial u_1}{\partial x_i}|^{p_i} \le C_7, \quad (21)$$

for some constant $C_7 > 0$. Moreover using (21) in (19) and iterating on s, we have

$$\sum_{i=1}^{N}\int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i}$$

$$\le C_8\Big[\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} + 1 + \sum_{i=1}^{N} A^{\frac{1}{p_i}-\frac{1}{p_\infty}} d_s^{\frac{p_i}{N}}\Big]$$

then arguing as before, we obtain $\sum_{i=1}^{N}\int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i} \le C_9$, for some constant $C_9 > 0$, then

$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le k\sum_{i=1}^{N}\Big(\sum_{s=1}^{t}\int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}} \le C_{10}$, for some positive $k > 0$.

Since u_n is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ and the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ is compact, we obtain the following results.

Corollaire 1 *If u_n is a weak solution of problem (13), then there exists a subsequence $(u_n)_n$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$, strongly in $L^{p_-}(\Omega)$ and a. e. in Ω.*

4.2 Strong convergence of $T_k(u_n)$

Lemma 2 *Assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$ and a. e. in Ω and $\sum_{i=1}^{N}\int_\Omega [a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u)](\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}) \to 0$ then,*

$$u_n \to u \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega).$$

Proof: The proof follows as in Lemma 5 of [15] taking into account the anisotropy of operator. □

Proposition 3 *Let u_n be a solution to the approximate problem (13), then*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega)$$

.

Proof: Let us fix k and let δ be a real number such that $\delta \ge (\frac{L(k)}{2\lambda})^2$. Let us define $z_n = T_k(u_n) - T_k(u)$ and $\varphi(s) = se^{\delta s^2}$, it is easy to check that for all $s \in \mathbb{R}$ one has

$$\varphi'(s) - \frac{L(k)}{\lambda}|\varphi(s)| \ge \frac{1}{2}. \quad (22)$$

Using $\varphi(z_n)$ as test function in (13), we get

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\frac{\partial \varphi(z_n)}{\partial x_i} + \sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)\varphi(z_n) + \sum_{i=1}^{N}\int_\Omega H_i^n(x,\nabla u_n)\varphi(z_n) =$$

$$\int_\Omega f\varphi(z_n) + \sum_{i=1}^{N}\int_\Omega k_i(x)\frac{\partial \varphi(z_n)}{\partial x_i}. \quad (23)$$

Since $\varphi(z_n) \rightharpoonup 0$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p_\infty}(\Omega)$ and $f \in L^{p'_\infty}(\Omega)$ then $\int_\Omega f\varphi(z_n) \to 0$ as $n \to +\infty$. Since $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$, $k_i \in L^{p'_i}(\Omega)$ and $(\varphi'(z_n))_n$ is bounded then $\sum_{i=1}^{N}\int_\Omega k_i(x)\frac{\partial \varphi(z_n)}{\partial x_i} \to 0$ as $n \to +\infty$. On the other hand, we have

$$|\sum_{i=1}^{N}\int_\Omega H_i^n(x,\nabla u_n)\varphi(z_n)| \le \sum_{i=1}^{N}\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i-1}|b_i\varphi(z_n)|$$

$$\le \sum_{i=1}^{N}\Big(\int_\Omega |b_i\varphi(z_n)|^{p_i}\Big)^{\frac{1}{p_i}}\Big(\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p'_i}}$$

$$\le C\sum_{i=1}^{N}\Big(\int_\Omega |b_i\varphi(z_n)|^{p_i}\Big)^{\frac{1}{p_i}}.$$

Since $b_i\varphi(z_n) \to 0$ a.e. in Ω and $|b_i\varphi(z_n)| \le |b_i| \times 2ke^{4k^2\delta} \in L^{p_i}(\Omega)$, then by dominated convergence theorem $b_i\varphi(z_n) \to 0$ strongly in $L^{p_i}(\Omega)$, then $\Big|\sum_{i=1}^{N}\int_\Omega H_i^n(x,\nabla u_n)\varphi(z_n)dx\Big| \to 0$ as $n \to +\infty$. Denote by $\varepsilon_1(n), \varepsilon_2(n), ...$ various sequences of real numbers which converge to zero when n tends to $+\infty$. Using (8), we obtain that $g_i^n(x,u_n,\nabla u_n)\varphi(z_n) \ge 0$ on the set $\{|u_n| > k\}$. Then we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\frac{\partial \varphi(z_n)}{\partial x_i} + \sum_{i=1}^{N}\int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)dx \le \varepsilon_1(n) \quad (24)$$

On the other hand, we have

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)$$
$$-\sum_{i=1}^{N}\int_{\{u_n>k\}} a_i(x,u_n,\nabla u_n)\frac{\partial T_k(u)}{\partial x_i}\varphi'(z_n).$$

The sequence $\Big(a_i(x,u_n,\nabla u_n)\varphi'(z_n)\Big)_n$ is bounded in $L^{p_i'}(\Omega)$, then since $\frac{\partial T_k(u)}{\partial x_i}\chi_{\{|u_n|>k\}}\to 0$ strongly in $L^{p_i}(\Omega)$, one has

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)+\varepsilon_2(n)$$

which we can write

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)dx+\varepsilon_2(n).$$

Since $u_n\to u$ a.e. in Ω, we have $a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)$ converges to $a_i(x,T_k(u),\nabla T_k(u))$ a. e. in Ω. Let E be measurable subset of Ω, by the growth condition (6), we get

$$\int_E |a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)|^{p_i'}dx$$
$$\le 2^{p_i'-1}(1+8\delta k^2)^{p_i'}e^{4\delta k^2 p_i'}\Big(k^{p_\infty}|E|+\int_E|\frac{\partial T_k(u)}{\partial x_i}|^{p_i}\Big).$$

Then the sequence $\Big(a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)\Big)_n$ is equi-integrable and by Vitali's theorem one has $a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)$ converges to $a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$. Since $\frac{\partial T_k(u_n)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$, then

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)=0.$$ It follows that

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}$$
$$=\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)+\varepsilon_3(n). \quad (25)$$

On the other hand, by virtue of (5) and (9)

$$|\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)dx|$$
$$\le L(k)\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|\frac{\partial u_n}{\partial x_i}|^{p_i}|\varphi(z_n)|dx$$
$$\le L(k)\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k u_n}{\partial x_i}|^{p_i}|\varphi(z_n)|dx$$
$$\le\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n)).\frac{\partial T_k(u_n)}{\partial x_i}|\varphi(z_n)|dx$$
$$\le\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|dx+$$
$$\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n)).\frac{\partial T_k(u)}{\partial x_i}|\varphi(z_n)|dx+$$
$$\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u)).\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|dx.$$

Similarly as above, it's easy to see that by (6), corollary 1 and Vitali's theorem one has $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$. Writing

$$\Big|\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u)).\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi(z_n)dx\Big|$$
$$\le\varphi(2k)\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|.\Big|\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big|$$

and taking into account that $\frac{\partial T_k(u_n)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u)).\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi(z_n)dx$$

tends to zero as $n\to+\infty$. Thanks to (6) and (14), the sequence $\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))\Big)_n$ is bounded in $L^{p_i'}(\Omega)$, so that there exists $l_k^i\in L^{p_i'}(\Omega)$ such that $a_i(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup l_k^i$ weakly in $L^{p_i'}(\Omega)$. We have

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx=$$
$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-l_k^i\Big)\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} l_k^i\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx.$$

Since $\varphi(z_n)\to 0$ weak* in $L^\infty(\Omega)$, we conclude that $\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx\to 0$ as $n\to+\infty$. Hence, we get

$$|\sum_{i=1}^{n}\int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)dx|$$

$$\le \frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) \qquad (26)$$

$$-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\tfrac{\partial T_k(u_n)}{\partial x_i}-\tfrac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|dx$$
$+\varepsilon_5(n)$.

Combining (24), (25) and (26), we obtain

$$\sum_{i=1}^{N}\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\Big(\varphi'(z_n)-\frac{L(k)}{\lambda}|\varphi(z_n)|\Big)dx\le\varepsilon_6(n).$$

Which gives by using (22)

$$0\le\sum_{i=1}^{N}\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\le 2\varepsilon_6(n),$$

then lemma 2 gives

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,\vec{p}}(\Omega). \qquad (27)$$

4.3 Existence results

Theorem 1 *Assume that $\overline{p}<N$ and (5)-(12) hold. Then there exists at least a weak solution of the problem (1).*

Proof: By (4.2) the sequence $(\frac{\partial u_n}{\partial x_i})_n$ is bounded in $L^{p_i}(\Omega)$, so we have that $\frac{\partial u_n}{\partial x_i}\rightharpoonup\frac{\partial u}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$ for $i=1,\dots,N$ and $u_n\to u$ strongly in $L^{p_-}(\Omega)$. By (27) there exists a subsequence, which we still denote by u_n such that $\frac{\partial u_n}{\partial x_i}\to\frac{\partial u}{\partial x_i}$ a. e. in Ω for $i=1,\dots,N$, then for $i=1,\dots,N$, we have

$$\begin{cases} a_i(x,u_n,\nabla u_n)\to a_i(x,u,\nabla u) \text{ a. e. in } \Omega,\\ g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u) \text{ a. e. in } \Omega,\\ H_i^n(x,\nabla u_n)\to H_i(x,\nabla u) \text{ a. e. in } \Omega.\end{cases}$$

Moreover by (6) and (10), we have

$$\int_\Omega|a_i(x,u_n,\nabla u_n)|^{p_i'}\le C\Big[\int_\Omega|u_n|^{p_\infty}+\int_\Omega|\frac{\partial u_n}{\partial x_i}|^{p_i}\Big] \quad\text{and}$$
$$\int_\Omega|H_i^n(x,\nabla u_n)|^{p_i'}\le C\int_\Omega|\frac{\partial u_n}{\partial x_i}|^{p_i}$$

by (14), $\Big(a_i(x,u_n,\nabla u_n)\Big)_n$ and $\Big(H_i(x,\nabla u_n)\Big)_n$ are bounded in $L^{p_i'}(\Omega)$ then $a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u)$ weakly in $L^{p_i'}(\Omega)$ and $H_i(x,\nabla u_n)\rightharpoonup H_i(x,\nabla u)$ weakly in $L^{p_i'}(\Omega)$. Now we prove that $g_i^n(x,u_n,\nabla u_n)$ is uniformly equi-integrable for $i=1,\dots,N$. For any measurable E of Ω and for any $k\in\mathbb{R}^+$, we have

$$\int_E|g_i^n(x,u_n,\nabla u_n)|$$
$$=\int_{E\cap\{|u_n|\le k\}}|g_i^n(x,u_n,\nabla u_n)|+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|$$
$$\le\int_{E\cap\{|u_n|\le k\}}L(k)|\frac{\partial T_k(u_n)}{\partial x_i}|^{p_i}+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|,$$

for fixed k and for $i=1,\dots,N$. For the first term we recall $\frac{\partial T_k(u_n)}{\partial x_i}$ strongly converges to $\frac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i}(\Omega)$ for $i=1,\dots,N$. Taking $T_k(u_n)$ as test function in (4.1), then

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\frac{\partial T_k(u_n)}{\partial x_i}+\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_k(u_n)+$$
$$\sum_{i=1}^{N}\int_\Omega H_i^n(x,\nabla u_n)T_k(u_n)=\int_\Omega fT_k(u_n)+\sum_{i=1}^{N}\int_\Omega k_i\frac{\partial T_k(u_n)}{\partial x_i},$$

which implies that

$$\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_k(u_n)\le\Big(\int_\Omega|f|^{p_\infty'}\Big)^{\frac{1}{p_\infty'}}\Big(\int_\Omega|u_n|^{p_\infty}\Big)^{\frac{1}{p_\infty}}+$$
$$\sum_{i=1}^{N}\Big(\int_\Omega|k_i|^{p_i'}\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega|\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}+$$
$$\sum_{i=1}^{N}\int_\Omega b_i|\frac{\partial u_n}{\partial x_i}|^{p_i-1}|T_k(u_n)|$$
$$\le\Big(\int_\Omega|f|^{p_\infty'}\Big)^{\frac{1}{p_\infty'}}\Big(\int_\Omega|u_n|^{p_\infty}\Big)^{\frac{1}{p_\infty}}+$$
$$\sum_{i=1}^{N}\Big(\int_\Omega|k_i|^{p_i'}\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega|\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}+$$
$$\sum_{i=1}^{N}\Big(\int_\Omega b_i^{r_i}\Big)^{\frac{1}{r_i}}\Big(\int_\Omega|\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega|u_n|^{p_\infty}\Big)^{\frac{1}{p_\infty}}$$
$$\le C_1,$$

where r_i is such that $\frac{1}{r_i}=\frac{1}{p_i}-\frac{1}{p_\infty}$. Let E a measurable subset of Ω and for any $m\in\mathbb{R}^+$, we have

$$\int_E|g_i^n(x,u_n,\nabla u_n)|dx$$
$$=\int_{E\cap\{|u_n|\le k\}}|g_i^n(x,u_n,\nabla u_n)|dx+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|dx$$
$$\le\int_{E\cap\{|u_n|\le k\}}L(k)\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i}dx+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|dx$$
$$\le\int_{E\cap\{|u_n|\le k\}}L(k)\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^{p_i}dx+\frac{1}{k}\int_{\{|u_n|>k\}}T_k(u_n)g_i^n(x,u_n,\nabla u_n)dx$$

we have $\frac{\partial T_k(u_n)}{\partial x_i}\to\frac{\partial T_k(u)}{\partial x_i}$ strongly in $L^{p_i}(\Omega)$ and

$$\int_{\{|u_n|>k\}}T_k(u_n)g_i^n(x,u_n,\nabla u_n)dx\le C_1$$

then g_i^n is uniformly equi-integrable for any i, since $g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)$ a. e. in Ω, we get $g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)$ in $L^1(\Omega)$. That allow us to pass to the limit in the approximate problem.

Remark 1 *The condition (6) can be substituted by $|a_i(x,s,\xi)|\le\gamma\Big[j_i+|s|^{\frac{p_\infty}{p_i'}}+|\xi_i|^{p_i-1}\Big]$, where j_i is a positive function in $L^{p_i'}(\Omega)$ for $i=1,\dots,N$, and the condition (9) can be substituted by $|g_i(x,s,\xi)|\le L(|s|)\Big(C_i+|\xi_i|^{p_i}\Big)$ where C_i is a positive function in $L^1(\Omega)$ for $i=1,\dots,N$.*

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