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Nonresonance between the first two Eigencurves of Laplacian for a Nonautonomous Neumann Problem

Ahmed Sanhaji*, Ahmed Dakkak

Sidi Mohamed Ben Abdellah University, Mathematics Physics and Computer Science, LSI, FP, Taza, Morocco

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ABSTRACT

We consider the following Neumann elliptic problem

$$\begin{cases} -\Delta u = \alpha\, m_1(x)\, u + m_2(x)\, g(u) + h(x) & in\, \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & on\, \partial\Omega. \end{cases}$$

By means of Leray-Schauder degree and under some assumptions on the asymptotic behavior of the potential of the nonlinearity g, we prove an existence result for our equation for every given $h \in L^\infty(\Omega)$.

1 Introduction

Let Ω be a bounded domain of $I\!R^N$ ($N \geq 1$), with $C^{1,1}$ boundary and let ν be the outward unit normal vector on $\partial\Omega$.
D. Del Santo and P. Omari, have studied in [1] the Dirichlet problem

$$\begin{cases} -\Delta u = g(u) + h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

They have proved the existence of nontrivial weak solutions for this problem for every given $h \in L^p(\Omega)$ under some assumptions on the function g. In the case of Neumann elliptic problem J.-P. Gossez and P. Omari, have considered in [2] the following problem

$$\begin{cases} -\Delta u = g(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

They have shown the existence of weak solutions for this problem for every given $h \in L^\infty(\Omega)$ under some conditions on function g. A.Dakkak and A. Anane studied in [3] the existence of weak solutions for the problem

$$\begin{cases} -\Delta u = \lambda_2 m(x) u + g(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda_2 = \lambda_2(m)$ is the second eigenvalue of $-\Delta$ with weight m, with $m \in M^+(\Omega) = \{m \in L^\infty(\Omega) : \text{meas}(\{x \in \Omega : m(x) > 0\}) \neq 0\}$.

We investigate in the present work the following Neumann elliptic problem

$$(\mathcal{P}) \begin{cases} -\Delta u = \alpha\, m_1(x) u + m_2(x) g(u) + h(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where $-\Delta$ is the Laplacian operator. The functions $m_1, m_2 \in M^+(\Omega)$, $h \in L^\infty(\Omega)$, $g : I\!R \to I\!R$ is a continuous function and α is a real parameter such that $\alpha \geq \lambda_2(m_1)$ or $\alpha \leq \lambda\_2(m_1)$, with $\lambda\_2(m_1) = -\lambda_2(-m_1)$. By a solution of $(\mathcal{P})$ we mean a function $u \in H^1(\Omega) \cap L^\infty(\Omega)$, such that

$$\int_\Omega \nabla u \nabla w = \int_\Omega (\alpha\, m_1 u + m_2 g(u) + h) w$$

for every $w \in H^1(\Omega)$.

This paper is organized as follows. In section 2, we recall some results that we will use later. Section 3 is concerned with the existence of principal eigencurve of the Laplacian operator with Neumann boundary conditions. In section 4, we show a theorem of nonresonance between the first and second eigenvalue (see theorem 2). In section 5, we prove the nonresonance between the first two eigencurves for problem $(\mathcal{P})$.

*Corresponding Author: Ahmed Sanhaji, Laboratory-LSI- sidi Mohamed ben Abdellah University, polydisciplinary, Faculty of TAZA, MOROCCO, & ahmed.sanhaji1@usmba.ac.ma

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2 Preliminary

Let us briefly recall some properties of the spectrum of $-\Delta$ with weight and with Neumann boundary condition to be used later. Let be Ω a smooth bounded domain in $I\!R^N$ $(N \geq 1)$ and let $m \in M^+(\Omega)$. the eigenvalue problem is

$$\begin{cases} -\Delta u = \lambda\, m(x)u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

this spectrum contains a sequence of nonnegative eigenvalues $(\lambda_n)_{n>0}$ given by

$$\frac{1}{\lambda_n} = \frac{1}{\lambda_n(m)} = \sup_{K\in\Gamma_n} \min_{u\in K} \frac{\int_\Omega mu^2}{\int_\Omega |\nabla u|^2}, \tag{2}$$

where $\Gamma_n = \{K \subset S : \text{K is symetric, compact and } \gamma(K) \geq n\}$, S is the unit sphere of $H^1(\Omega)$ and γ is the genus function. This formulation can be found in [4], the sequence $(\lambda_n)_{n>0}$ verify:
i) $\lambda_n \to +\infty$ as $n \to +\infty$
ii) If m change its sign in Ω and $\int_\Omega m dx < 0$, then the first eigenvalue defined by

$$\lambda_1(m) = \inf\left\{ \int_\Omega |\nabla u|^2, u \in H^1(\Omega) / \int_\Omega mu^2 dx = 1 \right\} \tag{3}$$

it is known that $\lambda_1(m)$ is > 0, simple and the associated eigenfunction φ_1 can be chosen such that $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{H^1} = 1$ hold. Moreover $\lambda_1(m)$ is isolated in the spectrum, which allows to define the second positive eigenvalue $\lambda_2(m)$ as

$$\lambda_2(m) = \min\{\lambda \in I\!R : \lambda \text{ is eigenvalue and } \quad \lambda > \lambda_1(m)\} \tag{4}$$

it is also known that any eigenfunction associated to a positive eigenvalue different from $\lambda_1(m)$ changes sign in Ω.
iii) $\lambda_1(m)$ is strictly monotone decreasing with respect to m (i.e. $m \lneqq m'$ implies $\lambda_1(m) > \lambda_1(m')$).
Throughout this work, the functions m_1 and m_2 satisfies the following assumptions:

$$(\mathbf{A}) \qquad m_1, m_2 \in M^+(\Omega) \text{ and } ess\inf_\Omega m_2 > 0.$$

Proposition 1 *([3]). Let $m, m' \in M^+(\Omega)$.*

1. If $m \leq m'$, then $\lambda_2(m) \geq \lambda_2(m')$.

2. $\lambda_2 : m \to \lambda_2(m)$ is continuous in $(M^+(\Omega), \|.\|_\infty)$.

Proposition 2 *([3]). Let $(m_k)_k$ be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$. then $\lim_{k\to\infty} \lambda_2(m_k) = +\infty$ if and only if $m \leq 0$ almost everywhere in Ω.*

3 Existence of the second eigencurve of the $-\Delta$ with weighs in the Neumann case

The second eigencurve of the $-\Delta$ with weighs is defined as a set $\mathcal{C}_2$ of those $(\alpha, \beta) \in I\!R^2$ such that the following Neumann problem

$$\begin{cases} -\Delta u = \alpha\, m_1(x)u + \beta m_2(x)u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a nontrivial solution $u \in H^1(\Omega)$ (i.e. $\mathcal{C}_2 = \Big\{(\alpha, \beta) \in I\!R^2; \lambda_2(\alpha\, m_1 + \beta m_2) = 1\Big\}$), where m_1 and m_2 satisfies the condition $(\mathbf{A})$. For more details see [5, 6]. The purpose in this section is to study the following problem: For $\beta < 0$, we prove the existence and the uniqueness of reel $\alpha_2^+(\beta)$ such that $(\alpha_2^+(\beta), \beta) \in \mathcal{C}_2$). Given $m \in M^+(\Omega)$, we denote by $\Omega_m^+ = \{x \in \Omega; m(x) > 0\}$ and $\Omega_m^- = \{x \in \Omega; m(x) < 0\}$.

Remark 1 *Let $(\alpha, \beta) \in \mathcal{C}_2$.*
1. If $\alpha > \lambda_2(m_1)$, then we have $\beta < 0$.
2. If $meas(\Omega_m^-) > 0$ and $\alpha < \lambda\_2(m_1)$, we have $\beta < 0$.

Indeed, assume by contradiction if $\alpha > \lambda_2(m_1)$ and $\beta \geq 0$, then

$$\alpha\, m_1 \leq \alpha\, m_1 + \beta\, m_2,$$

using the monotony property of λ_2, we obtain

$$\lambda_2(\alpha\, m_1 + \beta\, m_2) \leq \lambda_2(\alpha\, m_1) = \frac{\lambda_2(m_1)}{\alpha} < 1,$$

since, $(\alpha, \beta) \in \mathcal{C}_2$, we have $\lambda_2(\alpha\, m_1 + \beta\, m_2) = 1$, thus necessarily $\beta < 0$. The proof of the second assertion is similar.

Theorem 1 *Let m_1, m_2 satisfy $(\boldsymbol{A})$, then we have:*

i) *For all $\beta < 0$, there exists $\alpha_2^+(\beta) > \lambda_2(m_1)$ such that $(\alpha_2^+(\beta), \beta) \in \mathcal{C}_2$.*

ii) *If $meas(\Omega_{m_1}^-) > 0$, then for all $\beta < 0$, there exists $\alpha_2^-(\beta) < \lambda\_2(m_1)$ such that $(\alpha_2^-(\beta), \beta) \in \mathcal{C}_2$.*

Proof To prove **i)**, we consider $\beta < 0$ and we define $\alpha_2^+(\beta)$ as follows

$$\frac{1}{\alpha_2^+(\beta)} = \sup_{K\in\Gamma_2} \inf_{u\in K} \frac{\int_\Omega m_1\, u^2}{\int_\Omega |\nabla u|^2 - \beta \int_\Omega m_1\, u^2} \tag{5}$$

by definition of $\alpha_2^+(\beta)$ and, using the fact that for any eigenfunction associated to $\lambda_2(m_1)$ changes sign in Ω, we obtain that there exists eigenfunction u which change sign in Ω such that

$$\int_\Omega \nabla u.\nabla v - \beta \int_\Omega m_2\, u\, v = \int_\Omega \alpha_2^+(\beta)\, m_1\, v, \tag{6}$$

for all $v \in H^1(\Omega)$, we deduce also that, if $w \in H^1(\Omega)$ is eigenfunction of operator $-\Delta(.) - \beta\, m_2(.)$ which change singe in Ω with the corresponding eigenvalue $\lambda > 0$, then $\lambda \geq \alpha_2^+(\beta)$. In view of the (6), we have

$$\int_\Omega \nabla u.\nabla v = \int_\Omega (\alpha_2^+(\beta)\, m_1 + \beta m_2\, u)v \quad \text{for all}\, v \in H^1(\Omega)$$

it follows that the real 1 is eigenvalue of $-\Delta$ with weight $(\alpha_2^+(\beta)+\beta m_2)$, since the corresponding eigenfunction u change singe in Ω, we conclude that

$$\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)\le 1. \quad (7)$$

On the other hand, using (5) for all $K\in\Gamma_2$, there exists $u_K\in K$ such that

$$\min_{u\in K}\frac{\int_\Omega m_1 u^2}{\int_\Omega|\nabla u|^2-\beta\int_\Omega m_1 u^2} = \frac{\int_\Omega m_1 u_K^2}{\int_\Omega|\nabla u_K|^2-\beta\int_\Omega m_2 u_K^2}\le\frac{1}{\alpha_2^+(\beta)},$$

it follows that

$$\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u_K^2}{\int_\Omega|\nabla u_K|^2}\le 1.$$

So that

$$\min_{u\in K}\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u^2}{\int_\Omega|\nabla u|^2} \le\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u_K^2}{\int_\Omega|\nabla u_K|^2}\le 1 \text{ for all } K\in\Gamma_2,$$

this implies

$$\sup_{K\in\Gamma_2}\min_{u\in K}\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u^2}{\int_\Omega|\nabla u|^2}\le 1.$$

Since

$$\frac{1}{\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)}=\sup_{K\in\Gamma_2}\min_{u\in K}\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u^2}{\int_\Omega|\nabla u|^2}$$

we deduce that

$$\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)\ge 1. \quad (8)$$

By combining (7) and (8), we obtain

$$\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)=1.$$

Let $\gamma>0$ such that $\lambda_2(\gamma m_1+\beta m_2)=1$, there exists eigenfunction ω change singe in Ω and

$$\int_\Omega\nabla u.\nabla\omega=\int_\Omega(\gamma m_1+\beta m_2 u)\omega \quad \forall\,\omega\in H^1(\Omega)$$

hence

$$\int_\Omega\nabla u.\nabla\omega-\int_\Omega\beta m_2 u\,\omega=\gamma\int_\Omega m_1\omega \quad \forall\,\omega\in H^1(\Omega) \quad (9)$$

from (9), we obtain that γ is eigenvalue of the operator $-\Delta(.)-\beta m_2(.)$ with weight m_1, since the eigenfunction ω change singe, we conclude that

$$\gamma\ge\alpha_2^+(\beta).$$

Assume by contradiction that $\gamma>\alpha_2^+(\beta)$, then

$$\frac{1}{\gamma}<\frac{1}{\alpha_2^+(\beta)}=\sup_{K\in\Gamma_2}\min_{u\in K}\frac{\int_\Omega m_1 u^2}{\int_\Omega|\nabla u|^2-\beta\int_\Omega m_2 u^2},$$

by the inequality above we deduce that there exists $K_0\in\Gamma_2$ such that

$$\frac{1}{\gamma}<\min_{u\in K_0}\frac{\int_\Omega m_1 u^2}{\int_\Omega|\nabla u|^2-\beta\int_\Omega m_2 u^2},$$

since K_0 is compact, we conclude that, there exists $u_0\in K_0$

$$\frac{1}{\gamma}<\frac{\int_\Omega m_1 u_0^2}{\int_\Omega|\nabla u_0|^2-\beta\int_\Omega m_2 u_0^2},$$

hence

$$1<\min_{u\in K_0}\frac{\int_\Omega(\gamma m_1+\beta m_2)u_0^2}{\int_\Omega|\nabla u_0|^2}$$

it follows that

$$1<\sup_{K\in\Gamma_2}\min_{u\in K_0}\frac{\int_\Omega(\gamma m_1+\beta m_2)u_0^2}{\int_\Omega|\nabla u_0|^2}=\frac{1}{\lambda_2(\gamma m_1+\beta m_2)}=1,$$

which gives a contradiction, thus we have $\gamma=\alpha_2^+(\beta)$.

4 Nonresonance between the first and second eigenvalue

In this section we are interesting to the study of the existence results for the following Neumann problem

$$(\mathcal{P}_2)\begin{cases}-\Delta u=\lambda_2 m_1(x)u+m_2(x)g(u)+h(x) & \text{in } \Omega\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases}$$

where $\lambda_2=\lambda_2(m_1)$ is the second eigenvalue of $-\Delta$ with weight m_1 under the Neumann boundary condition.

Lemma 1 *Let $m_1, m_2\in M^+(\Omega)$. Assume that (**A**) is verified, then there exists a unique real $c>0$ such that*

$$\lambda_1(\lambda_2 m_1-c\,m_2)=1 \quad (10)$$

Proof Put $a=\dfrac{ess\inf_\Omega\lambda_2 m_1}{ess\inf_\Omega m_2}$ and $b=\dfrac{ess\sup_\Omega\lambda_2 m_1}{ess\inf_\Omega m_2}$ since m is a nonconstant function, then we have $a<b$. So for $t\in[a,b[$ we consider the weight $m_t=\lambda_2 m_1-t\,m_2$ and the corresponding increasing and continuous function:

$$f:[a,b[\to I\!R \qquad t\mapsto\lambda_1(\lambda_2 m_1-t\,m_2) \quad (11)$$

which satisfy $f(a)=0$ and $\lim_{t\to b}f(t)=+\infty$. According to be strict monotony property of λ_1 with weights we observe that f is strictly increasing on $[a',b[$ where $a'=\max\{t: f(t)=0\}$.
Therefore, $f(a')=0\le f(0)=\lambda_1(\lambda_2 m_1)<\lambda_2(\lambda_2 m_1)=1$, so the conclusion follows from the intermediate values theorem.

Theorem 2 *Let $m_1, m_2 \in M^+(\Omega)$. Assume that the weights m_1 and m_2 satisfy ($\boldsymbol{A}$) and the function g satisfy the following hypotheses*

$$-c \le \liminf_{s\to\pm\infty} \frac{g(s)}{s} \le \limsup_{s\to\pm\infty} \frac{g(s)}{s} \le 0 \tag{12}$$

$$-c < \limsup_{|s|\to\infty} \frac{2G(s)}{s^2}; \liminf_{s\to+\infty\, or -\infty} \frac{2G(s)}{s^2} < 0 \tag{13}$$

where c is given in (10), then problem ($\mathcal{P}_2$) admits at least one solution for any $h \in L^\infty(\Omega)$.

For the proof of theorem 2, we observe that the main trick introduce in [7] can be adapted in our situation. Furthermore the proof needs some technical lemmas, the two next lemmas concern an a-priori estimates on the possible solutions of the following homotopic problem.

$$\begin{cases} -\Delta u = ((1-\mu)\theta + \mu\lambda_2)m_1 u + \mu m_2 g(u) + \mu h & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where $\mu \in [0,1]$ and $\theta \in]\lambda_1, \lambda_2[$ to be variable and $\lambda_1 = \lambda_1(m_1)$.

Lemma 2 *Suppose that ($\boldsymbol{A}$) and (12) hold and assume that for some $\theta \in]\lambda_1, \lambda_2[$ there exists $\mu_{n,\theta} \in [0,1]$ and $u_{n,\theta}$ be a solution of (14), for all n. Then we have*

1) $(u_{n,\theta})_n$ is a sequence of $L^\infty(\Omega)$ and if $\|u_{n,\theta}\|_\infty \to +\infty$ when $n \to +\infty$, then

$$v_{n,\theta} = \frac{u_{n,\theta}}{\|u_{n,\theta}\|_\infty} \to v_\theta \text{ stongly in } C^1(\Omega), \tag{15}$$

for some subsequence.
2) Assume that the following hypothesis holds

$$(H) \quad \exists \theta \in]\lambda_1, \lambda_2[/ \limsup_{n\to\infty} \mu_{n,\theta} = 1$$

then one of the following assertions i) or ii) holds, where

i) $v_\theta = \psi$, ψ is a normed ($\|\psi\|_\infty = 1$) eigenfunction associated to $\lambda_2(m_1)$ and

$$\int_\Omega \frac{|g(u_{n,\theta})|}{\|(u_{n,\theta}\|} dx \longrightarrow 0 \text{ when } n \to +\infty. \tag{16}$$

Furthermore, there exists $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\eta_1 < \frac{\max(u_{n,\theta})}{-\min(u_{n,\theta})} < \eta_2 \text{ for } n \text{ large enough.} \tag{17}$$

ii) $v_\theta = \pm\varphi$, φ is a normed ($\|\varphi\|_\infty = 1$) eigenfunction associated to $\lambda_1(\lambda_2 m_1 - c m_2) = 1$ and

$$\int_\Omega \frac{|g(u_{n,\theta}) + c u_{n,\theta}|}{\|(u_{n,\theta}\|} dx \longrightarrow 0 \text{ when } n \to +\infty. \tag{18}$$

Furthermore, $u_{n,\theta}$ not changes sign for n large enough.

*3) If **(H)** is false, then there exists a sequence $(\theta_k)_k \subset]\lambda_1, \lambda_2[$ and a strictly increasing sequence $(n_k)_k \subset \mathbb{N}$ such that*

a) $\lim_{k\to+\infty} \theta_k = \lambda_2$, $\lim_{k\to\infty} \mu_{n_k,\theta_k} = 1$ and $\lim_{k\to\infty} \|w_k\| = +\infty$ where $w_k = u_{n_k,\theta_k}$.

b) $\frac{w_k}{\|w_k\|} \longrightarrow \pm\varphi$ strongly in $C^1(\overline{\Omega})$ and

$$\int_\Omega \frac{|g(w_k) + c w_k|}{\|w_k\|} dx \longrightarrow 0 \text{ when } n \to +\infty.$$

Proof: 1) From the Anane's L^∞-estimation [8] and the Tolksdorf's-regularity [9] we can see that $(u_{n,\theta})_n \subset \mathcal{C}^{1,\alpha}(\overline{\Omega})$, since the embedding $\mathcal{C}^{1,\alpha}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$ is continuous for some $\alpha \in]0,1[$ independent on n, furthermore $v_{n,\theta} = \frac{u_{n,\theta}}{\|u_{n,\theta}\|_\infty}$ remains a bounded sequence in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.
By using the following compact embedding $\mathcal{C}^{1,\alpha}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}^1(\overline{\Omega})$, then there exists a subsequence still denoted $(v_{n,\theta})_n$ such that

$$v_{n,\theta} \to v_\theta \text{ stongly in } \mathcal{C}^1(\overline{\Omega}) \text{ and } \|v_\theta\|_\infty = 1. \tag{19}$$

2) According to the function g satisfy the hypothesis (12), we deduce that for all $s \in \mathbb{R}$, we can write

$$g(s) = q(s)s + r(s) \tag{20}$$

where $-c \le q(s) \le 0$ and $\frac{r(s)}{s} \longrightarrow 0$ uniformly, when $|s| \to +\infty$. Since $u_{n,\theta}$ is a solution of $(\mathcal{P}_{2,\theta,\mu_n})$, we get

$$\begin{aligned} \int_\Omega \nabla u_{n,\theta} \nabla w dx = & \int_\Omega [(1-\mu_n)\theta + \mu_n\lambda_2) m_1 u_{n,\theta} \\ & + \mu_n m_2 g(u_{n,\theta}) + \mu_n h] w dx \end{aligned} \tag{21}$$

for all $w \in H^1(\Omega)$.
On the other hand, since $(u_{n,\theta})_n \subset L^\infty(\Omega)$ and q is a continuous function, it follows that $q(u_n)$ is bonded in $L^\infty(\Omega)$, then for a subsequence we get

$$q(u_{n,\theta}) \rightharpoonup q_\theta \text{ in } L^\infty(\Omega) \text{ weak} - *,$$

and

$$\frac{|r(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} \to 0 \text{ strongly in } L^\infty(\Omega)$$

where $-c \le q_\theta(x) \le 0$ a.e. in Ω.
Dividing by $\|u_{n,\theta}\|_\infty$ and passing to the limit as $n \to \infty$ in (20), we get

$$\int_\Omega \nabla v_\theta \nabla w dx = \int_\Omega (\lambda_2 m_1 + q_\theta m_2) v_\theta w dx \ \forall w \in H^1(\Omega). \tag{22}$$

Since $v_\theta \neq 0$, then 1 is an eigenvalue of Laplacain with weight $m_{q_\theta} = \lambda_2 m_1 + q_\theta m_2$.
By using the monotony property of λ_2 with respect to the weight, we obtain

$$\lambda_2(m_{q_\theta}) \ge \lambda_2(\lambda_2 m_1) = 1$$

hence $\lambda_2(m_{q_\theta}) = 1$ or $\lambda_2(m_{q_\theta}) > 1$.
First case: $\lambda_2(m_{q_\theta}) = 1$.
So, we have

$$q_\theta = 0 \quad \text{and} \quad v_\theta = \psi.$$

Moreover, let us denotes by F_{λ_2} be the eigenspace associated to $\lambda_2 = \lambda_2(m_1)$, since F_{λ_2} is a vector space of finite dimension then, we can take

$$\eta_1 < \min_{v \in F_{\lambda_2} \cap S} \frac{\max(v)}{-\min(v)} \quad \text{and} \quad \eta_2 > \max_{v \in F_{\lambda_2} \cap S} \frac{\max(v)}{-\min(v)}$$

where $S = \left\{ v \in F_{\lambda_2} / \|v\|_\infty = 1 \right\}$.
It is clear to see that $\frac{\max(v_{n,\theta})}{-\min(v_{n,\theta})} \to \frac{\max(\psi)}{-\min(\psi)}$ when $n \to +\infty$.
It follows that for n large enough

$$\eta_1 < \frac{\max(v_{n,\theta})}{-\min(v_{n,\theta})} < \eta_2.$$

According to (20), we get

$$\begin{aligned} \int_\Omega \frac{|m_2(x) g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx &\le \|m_2\|_\infty \int_\Omega -q(u_{n,\theta}) |v_{n,\theta}| \\ &+ \|m_2\|_\infty \int_\Omega \frac{|r(u_{n\theta})|}{\|u_{n\theta}\|_\infty} dx \end{aligned}$$

by passage to the limit in the above inequality, we find (16).
Second case: $\lambda_2(m_{q_\theta}) > 1$.
So, we get $\lambda_1(m_{q_\theta}) = 1$ and by using the strict monotony property of λ_1, we can see that

$$q_\theta = -c \quad \text{and} \quad v_\theta = \varphi.$$

The rest of the proof follows directly as in first case.
3) We take $(\theta_k)_k \subset]\lambda_1, \lambda_2[$ such that $\lim_{k \to +\infty} \theta_k = \lambda_2$. Let us denotes by $\overline{\mu}_k = \lim_{n \to +\infty} \mu_{k,\theta_k}$, as in 2) it is clear to see that

$$\begin{cases} -\Delta u = [((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1 + \overline{\mu}_k q_{\theta_k} m_2] v_{\theta_k} & \text{in } \Omega \\ \frac{\partial v_{\theta_k}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

we recall that $0 \le \overline{\mu}_k < 1$, $-c \le q_{\theta_k} \le 0$ and $q(u_{n,\theta_k}) \longrightarrow^* q_{\theta_k}$ in $L^\infty(\Omega)$ when $n \to +\infty$, where $g(s) = q(s)s + r(s)$, $-c \le q(s) \le 0$ and $\frac{r(s)}{r} \to 0$ when $|s| \to +\infty$.
Let us consider the following weight

$$\overline{M}_k(x) = ((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1(x) + \overline{\mu}_k q_{\theta_k} m_2(x)$$

we have

$$\overline{M}_k(x) \le ((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1(x)$$

because $q_{\theta_k} \le 0$.
Then

$$\begin{aligned} \lambda_2(\overline{M}_k(x)) &\ge \lambda_2(((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1(x)) \\ &\ge \frac{\lambda_2}{(1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2} > 1. \end{aligned}$$

Thus it is clear to see that

$$\lambda_2(\overline{M}_k(x)) = 1. \tag{23}$$

Let $\overline{\mu}_k \to \overline{\mu}$ and $q(u_{n,\theta_k}) \longrightarrow^* \overline{q}_0$ in $L^\infty(\Omega)$ when $k \to +\infty$, where $-c \le \overline{q}_0 \le 0$. Then by letting k tends to infinity in (23), it follows that

$$\lambda_1(\lambda_2 m_1 + \overline{\mu}\,\overline{q}_0 m_2) = 1. \tag{24}$$

In view of lemma 1, we get

$$\overline{\mu} = 1, \ \overline{q}_0 = -c \quad \text{and } u_{\theta_k} \to \pm\varphi \text{ strongly in } C^1(\Omega) \text{ when } k \to +\infty.$$

On the other hand, using (20), we have

$$\begin{aligned} \int_\Omega \frac{|g(w_k) + c w_k|}{\|w_k\|_\infty} dx &= \int_\Omega \frac{|q(w_k) w_k + c w_k + r(w_k)|}{\|w_k\|_\infty} dx \\ &\le \int_\Omega |q(w_k) + c| \frac{w_k|}{\|w_k\|_\infty} dx \\ &+ \int_\Omega \frac{|r(w_k)|}{\|w_k\|_\infty}. \end{aligned}$$

Since $\frac{r(s)}{s} \to 0$ when $|s| \to \infty$, then for all $\varepsilon > 0$ there exists $n_{\varepsilon,k}$ such that for all $n \ge n_{\varepsilon,k}$

$$\int_\Omega \frac{|g(w_k) + c w_k|}{\|w_k\|_\infty} dx \le \int_\Omega |\overline{q}_{\theta_k} + c| |v_{\theta_k}| dx + \varepsilon.$$

If we take $\varepsilon = \frac{1}{k}, n = n_k = n_{\varepsilon,k}$ and we replace in the last formula, then we can see that the second member of this last inequality goes to 0 when m goes to $+\infty$. Finally, we conclude by the fact that n_k will be adjusted such that $\|u_{n_k,\theta_k}\|_\infty > k$ and $\|v_{n_k,\theta_k} - v_{\theta_k}\|_{C^1(\overline{\Omega})} \le \frac{1}{k}$.

Lemma 3 *Let us consider the assumptions and notations of lemma 2. We take $a \in \Omega$ and $\eta > 0$ such that $B(a,\eta) \subset \Omega$.*

*1) Assume that the hypothesis (**H**) holds, so, if $\|u_{n,\theta}\|_\infty \to +\infty$ when $n \to +\infty$ then $\limsup_{n \to +\infty} \mu_{n,\theta} = 1$ and*
i) If $v_\theta = \psi$ then

$$\lim_{n \to \infty} \int_0^1 \frac{|g(u_{n,\theta}(\sigma_x(t)))| |\nabla u_{n,\theta}(\sigma_x(t))| |x-a|}{\|u_{n,\theta}\|_\infty} dt = 0 \quad a.e.\ x \in \partial B(a,\eta) \tag{25}$$

where $\sigma_x(t) = a + t(x-a)$.
ii) If $v_\theta = \pm\varphi$ then

$$\lim_{n \to \infty} \int_0^1 \frac{|g(u_{n,\theta}(\sigma_x(t))) + c u_{n,\theta}(\sigma_x(t)| |\nabla u_{n,\theta}(\sigma_x(t))| |x-a|}{\|u_{n,\theta}\|_\infty} dt = 0 \quad a.e.\ x \in \partial B(a,\eta). \tag{26}$$

*2) Assume that **(H)** is false then we have*

$$\lim_{n \to \infty} \int_0^1 \frac{|g(w_k(\sigma_x(t))) + c w_k(\sigma_x(t)| |\nabla w_k(\sigma_x(t))| |x-a|}{\|w_k\|_\infty} dt = 0 \quad a.e.\ x \in \partial B(a,\eta). \tag{27}$$

Proof: We only show the relation (25) since the proof of the (26) and (27) one proceeds in the same way. Firstly we have

$$\int_{B(a,\eta)} \frac{|g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx \le \int_\Omega \frac{|g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx.$$

According to (16), we obtain

$$\lim_{n \to \infty} \int_{B(a,\eta)} \frac{|g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx = 0.$$

Which gives by passing in spherical coordinates

$$\lim_{n\to\infty}\int_{[0,\pi]^n}\int_0^{2\pi}\int_0^{\eta} t^{N-1}\frac{|g(u_{n,\theta}(a+t\omega))|}{\| u_{n,\theta}\|_\infty} \prod_{j=1}^{N-2}(\sin\theta_j)^{N-1-j}d\theta_j d\theta_{N-1}dt = 0 \quad (28)$$

where, $\omega = \frac{x-a}{\eta} \in \partial B(0,1)$.
The above equality imply

$$\frac{|g(u_{n,\theta}(\sigma_x(\tau)))|}{\|u_{n,\theta}\|_\infty} \to 0 \quad \text{when } n\to\infty \quad \text{a.e. } x\in\partial B(a,\eta) \text{ and a.e. } \tau\in[0,1].$$

By using (12) we can see that g satisfy the following growth condition:

$|g(s)| \le a|s| + b$, for some positive reals a, b and $\forall s \in I\!R$

hence $\left(\frac{|g(u_{n,\theta}(\sigma_x(.)))|}{\|u_{n,\theta}\|_\infty}\right)_n$ and $\left(\frac{|\nabla(u_{n,\theta}(\sigma_x(.)))|}{\|u_{n,\theta}\|_\infty}\right)_n$ are bounded in $L^\infty([0,1])$.
By using the Lebesgue dominated convergence theorem, we conclude this proof.

Lemma 4 *1) If G satisfy* $-c < \limsup_{s\to+\infty}\frac{2G(s)}{s^2} \le 0$. *then for all* $r\in]0,1[$, *there exists* $\rho_r > -c(1-r^2)$ *and a sequence of positive real numbers* (S_n) *such that*

$$\lim_{n\to+\infty} S_n = +\infty \textit{ and } \lim_{n\to+\infty}\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho_r.$$

2) If G satisfy $-c < \limsup_{s\to+\infty}\frac{2G(s)}{s^2} \le 0$ *and* $-c < \liminf_{s\to+\infty}\frac{2G(s)}{s^2} < 0$, *then for all* $r\in]0,1[$, *there exists* $\rho_r > -c(1-r^2)$ *and a sequence of positive real numbers* $(S_n^{'})$ *such that*

$$\lim_{n\to+\infty} S_n^{'} = +\infty, \quad \lim_{n\to+\infty}\frac{2G(S_n^{'})}{S_n^{'2}} = \rho \quad \textit{and } \lim_{n\to+\infty}\frac{2G(S_n^{'})-2G(rS_n^{'})}{S_n^{'2}} = \rho_r.$$

The same conclusion is obtained when we replace $+\infty$ par$-\infty$, in this case we should note $\rho^{'}$ and $\rho_r^{'}$ in place of ρ and ρ_r respectively.

Proof: We pose $L_1 = \liminf_{s\to+\infty}\frac{2G(s)}{s^2}$ and $L_2 = \limsup_{s\to+\infty}\frac{2G(s)}{s^2}$.

We distinguish the following two cases:
Case1: If $L_1 = L_2$, it is enough to take $\rho = L_2$ and $\rho_r = (1-r^2)L_2$.
Case2: If $L_1 < L_2$, then we choose $\rho\in]L_1, L_2[$ neighbor of L_2 in such a way that:

$$\rho - r^2L_2 > -c(1-r^2).$$

According to the definition of L_1 and L_2, we conclude that, there existence a sequence (S_n) such that $\lim_{n\to+\infty} S_n = +\infty$ and $\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho$. We put

$$\liminf_{n\to+\infty}\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho - r^2\limsup_{n\to+\infty}\frac{2G(rS_n)}{rS_n^2}.$$

Since $\limsup_{n\to+\infty}\frac{2G(rS_n)}{(rS_n)^2} \le L_2$, then we have:

$$\rho_r \ge \rho - r^2L_2 > -c(1-r^2).$$

5 Proof of Theorem 2

Our purpose now, consists in building in $C(\overline{\Omega})$ an open bounded set $\mathcal{O}$ such that, there exist $\theta\in]\lambda_1,\lambda_2[$ such that, no solution of (14) with $\mu\in[0,1[$ occurs on the boundary $\partial\mathcal{O}$. Homotopy invariance of the degree then yields the conclusion. The set $\mathcal{O}$ will have the following form

$$\mathcal{O} = \mathcal{O}_{S,T} = \{u\in \mathcal{C}(\overline{\Omega}); \quad T < u < S\},$$

where, S and T satisfy $T < 0 < S$.
Firstly, according to, (13) we will assume the following hypothesis holds

$$-c < \limsup_{|s|\to\infty}\frac{2G(s)}{s^2} \text{ and } \liminf_{s\to+\infty}\frac{2G(s)}{s^2} < 0 \quad (13_+).$$

An analogous proof will be adapted to the hypothesis (13_-) where (13) represents (13_+) or (13_-). Assume by contradiction, that for all $\theta\in]\lambda_1,\lambda_2[$ and for all S, T $(T < 0 < S)$, there exists $\mu = \mu_{\theta,S,T}\in[0,1[$ and $u = u_{\theta,S,T}\in\partial\mathcal{O}$ such that u is a solution of (14) which gives

$$\max(u) = S \text{ or } \min(u) = T. \quad (29)$$

According to (13_+), then owing to lemma 4, there exists two sequence (T_n) and (S_n) such that

$$\lim_{n\to+\infty} T_n = -\infty, \lim_{n\to+\infty}\frac{2G(T_n)-2G(rT_n)}{T_n^2} = \rho_r^{\prime} \quad (30)$$

$$\lim_{n\to+\infty} S_n = +\infty, \lim_{n\to+\infty}\frac{2G(S_n)}{S_n^2}, \lim_{n\to+\infty}\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho_r \quad (31)$$

where $r = \frac{\min\varphi}{\max\varphi}$ and ρ, ρ_r and $\rho_r^{\prime}$ provides from lemma 4.
So, we remark that, without loss of generality we can assume that

$$\frac{S_n}{-T_n} \le \eta_1. \quad (32)$$

We use the following notations:

$$T = T_n,\ S = S_n,\ \mu_{n,\theta} = \mu_{\theta,T,S} \text{ and } u_{n,\theta} = u_{\theta,T,S}.$$

According to (29) and (31), we obtain

$$\|u_{n,\theta}\|_\infty \to +\infty \text{ when } n\to+\infty.$$

We will distinguish two cases.
First case: We assume that the hypothesis **(H)** is satisfied (c.f. lemma 2).
According to lemma 2 we get

$$v_{n,\theta} = \frac{u_{n,\theta}}{\|u_{n,\theta}\|_\infty} \to v_\theta \text{ where } v_\theta = \psi \text{ or } v_\theta = \pm\varphi.$$

i) If $v_\theta = \psi$, we assert that for every n large enough we have

$$\max(u_{n,\theta}) = S_n.$$

Indeed, if not we have

$$\max(u_{n,\theta}) < S_n \text{ and } \min(u_{n,\theta}) = T_n$$

thus,

$$\frac{\max(u_{n,\theta})}{-\min(u_{n,\theta})} < \frac{S_n}{-S_n} \le \eta_1.$$

which gives a contradiction from the definition of η_1 given in lemma 2. On the other hand, we have $u_{n,\theta}$ changes sign on Ω, so there exists $x_n \in \overline{\Omega}$, $y_n \in \Omega$ such that

$$u_{n,\theta}(x_n) = S_n \text{ and } u_{n,\theta}(y_n) = 0,$$

we write

$$\frac{2G(S_n)}{S_n^2} = \frac{2G(u_{n,\theta}(x_n)) - 2G(u_{n,\theta}(x_n))}{\|u_{n,\theta}\|_\infty^2 \max(v_{n,\theta})^2} = \frac{2}{\max(v_{n,\theta})^2}\int_{\mathcal{C}_n} \frac{d(G \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2}$$

where

$$d(G \circ u_{n,\theta})(\mathcal{C}_n) = g(u_{n,\theta}(\mathcal{C}_n))\nabla u_{n,\theta}(\mathcal{C}_n).\mathcal{C}_n', \text{ a.e.}$$

and $\mathcal{C}_n$ is a $\mathcal{C}^1$ with morsels line which connects extremity x_n and y_n.
According to lemma 3, we have

$$\lim_{n\to+\infty}\int_{\mathcal{C}_n} \frac{d(G \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2}.$$

Since $\max(v_{n,\theta}) \to \max(\psi)$ when $n \to +\infty$, we deduce that

$$\rho = \lim_{n\to+\infty} \frac{2G(S_n)}{S_n^2} = 0,$$

which is a contradiction since $\rho \in]-c, 0[$.
ii) If $v_\theta = \pm\varphi$: then, for n large enough $u_{n,\theta}$ not changes sign on Ω.
so,

$$\max(u_{n,\theta}) = S_n \text{ or } \min(u_{n,\theta}) = T_n.$$

Assume that $\max(u_{n,\theta}) = S_n$ (the same gait will be used for the case $\min(u_{n,\theta}) = T_n$), so it is clear to see that $v_{n,\theta} = +\varphi$ and $\min(u_{n,\theta}) > 0$ for n large enough.
We put

$$\overline{g}(s) = g(s) + cs,\ \overline{G}(s) = \int_0^s \overline{g}(s)ds.$$

Let x_n, $y_n \in \overline{\Omega}$ such that $u_{n,\theta}(x_n) = S_n$ and $u_{n,\theta}(y_n) = \min(u_{n,\theta})$.
We write

$$\frac{\overline{G}(S_n) - \overline{G}(r_nS_n)}{S_n^2} = \frac{2\overline{G}(u_{n,\theta}(x_n)) - 2\overline{G}(u_{n,\theta}(x_n))}{\|u_{n,\theta}\|_\infty^2} = \int_{\mathcal{C}_n} \frac{d(\overline{G} \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2}$$

where $r_n = \frac{\min(u_{n,\theta})}{\max(u_{n,\theta})} \to r = \frac{\min(\varphi)}{\max(\varphi)}$.
Using the Lemma 3, we deduce that

$$\lim_{n\to+\infty}\int_{\mathcal{C}_n} \frac{d(\overline{G} \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2} = 0. \tag{33}$$

On the other hand, it is easy to verify that

$$0 = \lim_{n\to+\infty} \frac{\overline{G}(S_n) - \overline{G}(r_nS_n)}{S_n^2} = \frac{\rho_r + c(1-r^2)}{2} > 0$$

which gives a contradiction.
Second case: Assume that the hypothesis **(H)** is not verified, so for all $\theta \in]\lambda_1, \lambda_2[$ such that $\limsup_{n\to+\infty} \mu_{n,\theta} < 1$.
We take a sequence (θ_k) such that

$$\lim_{k\to+\infty} \theta_k = \lambda_2,$$

and we consider the subsequences

$$(T_{n_k})_k, (S_{n_k})_k \text{ and } w_k = u_{n_k,\theta_k}.$$

Similarly, as in the second point of the previous case we obtain a contradiction. This completes the proof of theorem 2.

6 Nonresonance between the first two Eigencurves

In this section we will prove an existence result for problem $(\mathcal{P})$. We need more restrictive hypotheses on the nonlinearities g and G.

$$(\mathbf{A}_g) \qquad \beta_1 \le \liminf_{s\to\pm\infty} \frac{g(s)}{s} \le \limsup_{s\to\pm\infty} \frac{g(s)}{s} \le \beta_2$$

$$(\mathbf{A}_G^\pm) \quad \beta_1 < \limsup_{|s|\to\infty} \frac{2G(s)}{s^2};\ \liminf_{s\to+\infty\, or\, -\infty} \frac{2G(s)}{s^2} < \beta_2$$

where $(\beta_1, \beta_2) \in I\!R^2$ with $\beta_2 - \beta_1 = c$ and c is given in (10).

Theorem 3 *Let m_1, $m_2 \in M^+(\Omega)$. Assume that $(\boldsymbol{A})$, $(\boldsymbol{A}_g)$ and $(\boldsymbol{A}_G^\pm)$ holds. Moreover, if $(\alpha, \beta_1) \in \mathcal{C}_1$ and $(\alpha, \beta_2) \in \mathcal{C}_2$, where $\alpha \ge \lambda_2(m_1)$ or $\alpha \le \lambda\_2(m_1)$. Then the problem $(\mathcal{P})$ has at least one nontrivial weak solution $u \in H^1(\Omega)$ for any given $h \in L^\infty(\Omega)$.*

Proof: The problem $(\mathcal{P})$ can be written in the following equivalent form

$$(\mathcal{P}_e) \quad \begin{cases} -\Delta u = \widetilde{m}|u|^{p-2}u + m_2\widetilde{g}(u) + h & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\widetilde{g}(s) = g(s) - \beta_2 s,$$

and

$$\widetilde{m} = \alpha\, m_1 + \beta_2\, m_2.$$

Since $(\alpha, \beta_2) \in \mathcal{C}_2$, then 1 is the second eigenvalue of laplacian operator with weight $\widetilde{m}$ relating to Neumann boundary conditions. In view of theorem 2, there exists at least one weak solution $u \in H^1(\Omega)$ of the problem $(\mathcal{P}_e)$ for all $h \in L^\infty(\Omega)$ if the function $\widetilde{g}$ and his potential $\widetilde{G}$ satisfy the two conditions (12) and (13). Indeed, in view of $(\mathbf{A}_g)$

$$\beta_1 - \beta_2 \leq \lim_{|s|\to\infty} \inf \left(\frac{g(s)}{s} - \beta_2 \right) \leq 0$$

thus

$$-c \leq \lim_{|s|\to\infty} \inf \left(\frac{\widetilde{g}(s)}{s} \right) \leq 0.$$

Consequently, $\widetilde{g}$ satisfy (12).
On the other hand, using $(\mathbf{A}_G^{\pm})$, we have

$$\beta_1 - \beta_2 < \limsup_{|s|\to\infty} \left(\frac{2G(s)}{s^2} - \beta_2 \right); \liminf_{s\to+\infty\, or -\infty} \left(\frac{2G(s)}{s^2} - \beta_2 \right) < 0$$

hence

$$-c < \limsup_{|s|\to\infty} \left(\frac{2\widetilde{G}(s)}{s^2} \right); \liminf_{s\to+\infty\, or -\infty} \left(\frac{2\widetilde{G}(s)}{s^2} \right) < 0$$

which means that $\widetilde{G}$ satisfy (13).
Finally, since the problem $(\mathcal{P}_e)$ is a equivalent to the problem $(\mathcal{P})$. Then the problem $(\mathcal{P})$ has at least one nontrivial weak solution $u \in H^1(\Omega)$ for any given $h \in L^\infty(\Omega)$.

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