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Existence and Boundedness of Solutions for Elliptic Equations in General Domains

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ABSTRACT

This article is devoted to study the existence of solutions for the strongly nonlinear $p(x)$-elliptic problem:

$$-\Delta_{p(x)}(u) + \alpha_0 |u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1} + f - \operatorname{div} g(x) \quad in\ \Omega,$$
$$u \in W_0^{1,p(x)}(\Omega),$$

where Ω is an open set of $\mathbb{R}^N$, possibly of infinite measure, also we will give some regularity results for these solutions.

1 Introduction

In recent years, there has been an increasing interest in the study of various mathematical problems with variable exponents. These problems are interesting in applications (see [[1], [2]]). For the usual problems when p is constant, there are many results for existence of solutions when the domain is bounded or unbounded. For p variable, when the domain is bounded, on the results of existence of solutions, we refer to [[3], [4], [5]], when the domain is unbounded, results of existence of solutions are rare we can cite for example [[6], [7]].

In the case where Ω is a bounded, and for $1 < p < N$, In [8] authors studied the problem:

$$-\operatorname{div} a(x,u,\nabla u) = H(x,u,\nabla u) + f - \operatorname{div} g \quad \text{in } D'(\Omega),$$
$$u \in W_0^{1,p}(\Omega),$$

where the right hand side is assumed to satisfy:

$$f \in L^{N/p}(\Omega), g \in (L^{N/(p-1)}(\Omega))^N.$$

Under suitable smallness assumptions on f and g they prove the existence of a solution u which satisfies a further regularity.

In [9] in the case of unbounded domains Guowei Dai By variational approach and the theory of the variable exponent Sobolev spaces establish the existence of infinitely many distinct homoclinic radially symmetric solutions whose $W^{1,p(x)}(\mathbb{R}^N)$-norms tend to zero (to infinity, respectively) under weaker hypotheses about nonlinearity at zero (at infinity, respectively).

The principal objective of this paper is to prove the existence and some regularity of solutions of the following p(x)-Laplacian equation in open set Ω of $\mathbb{R}^N$ (possibly of infinite measure):

$$-\Delta_{p(x)}(u) + \alpha_0 |u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1} + f - \operatorname{div} g(x) \quad \text{in } \Omega,$$
$$u \in W_0^{1,p(x)}(\Omega), \tag{1}$$

where p is log-Hölder continuous function such that $1 < p_- \le p_+ < N$, $\Delta_{p(x)}(u) = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator, α_0 is a positive constant, d is a function in $L^\infty(\Omega)$. We assume the following hypotheses on the source terms f and g :
$f : \Omega \to \mathbb{R}$, $g : \Omega \to \mathbb{R}^N$ are a measurable function

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satisfying:

$$\begin{gathered} f \in L^{N/p(x)}(\{x \in \Omega : 1 < |f(x)|\}), \\ f \in L^{p'(x)}(\{x \in \Omega : |f(x)| \leq 1\}) \\ g \in L^{N/(p(x)-1)}(\Omega;\mathbb{R}^N) \cap L^{p'(x)}(\Omega;\mathbb{R}^N) \end{gathered} \tag{2}$$

We will proceed by solving the problem on a sequence Ω_n of bounded sets after that we pass to the limit in the approximating problems by using the a priori estimate (this a priori estimates provide the necessary compactness properties for solutions) from which the desired results are easily inferred. To this aim, we can neither use any embedding theorem between $L^{p(.)}(\Omega)$ nor any argument involving the measure of Ω_n, and under suitable assumptions on f and g we prove some regularity of a solutions u of (1). A similar result has been proved in [7] where p is constant such that $1 < p < N$ but in the present setting such an approach cannot be used directly, because of the variability of p.

The plan of the paper is the following: In Section 2 we recall some important defnitions and results of variable exponent Lebesgue and Sobolev spaces. In Section 3 we will give the precise assumptions and state the main results. In Section 4 we will define the approximate problems, state the a priori estimates that we want to obtain. In the Sections which follow we will prove strong convergence of u_n and their gradients ∇u_n. Section 5 is devoted to conclude the proof of the main existence results. Finally, in Section 6, we prove that, if f and g have higher integrability, then every solution u of (1) is bounded. More precisely, we will assume that (2) are replaced by:

$$\begin{gathered} f \in L^{q(x)}(\{x \in \Omega : 1 < |f(x)|\}) \text{ for some } q(x) > N/p(x), \\ f \in L^{p'(x)}(\{x \in \Omega : |f(x)| \leq 1\}) \\ g \in L^{r(x)}(\Omega;\mathbb{R}^N) \cap L^{p'(x)}(\Omega;\mathbb{R}^N) \\ \text{for some } r(x) > N/(p(x)-1) \end{gathered} \tag{3}$$

2 Preliminaries

In order to discuss the problem (1), we need to recall some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents.

Let Ω an open bounded set of $\mathbb{R}^N$ with $N \geq 2$. We say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if:

$$|p(x)-p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x,y \in \overline{\Omega} \quad \text{such that } |x-y| < \frac{1}{2},$$

We denote:

$$C_+(\overline{\Omega}) = \{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p_- \leq p_+ < N\},$$

where:

$$p_- = ess\min_{x\in\overline{\Omega}} p(x) \quad p_+ = ess\sup_{x\in\overline{\Omega}} p(x).$$

We define the variable exponent Lebesgue space for $p \in C_+(\overline{\Omega})$ by:

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

the space $L^{p(x)}(\Omega)$ under the norm:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\Big\}$$

is a uniformly convex Banach space, and therefore reflexive.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [10, 11]).

Proposition 1 (Generalized Hölder inequality [10, 11]) *(i) For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$|\int_\Omega uv\,dx| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

(ii) For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that: $p_1(x) \leq p_2(x)$ a.e. in Ω, we have: $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2 ([10, 11]) *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)} dx \quad \forall u \in L^{p(x)}(\Omega),$$

then, the following assertions hold

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1$ *(resp, = 1, > 1) if and only if* $\rho(u) < 1$ *(resp, = 1, > 1);*

(ii) $\|u\|_{L^{p(x)}(\Omega)} > 1$ *implies* $\|u\|^{p_-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_+}_{L^{p(x)}(\Omega)}$, *and* $\|u\|_{L^{p(x)}(\Omega)} < 1$ *implies* $\|u\|^{p_+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_-}_{L^{p(x)}(\Omega)}$;

(iii) $\|u\|_{L^{p(x)}(\Omega)} \to 0$ *if and only if* $\rho(u) \to 0$, *and* $\|u\|_{L^{p(x)}(\Omega)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

Now, we define the variable exponent Sobolev space by:

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm:

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 3 ([10, 12]) *(i) If $1 < p_- \leq p_+ < \infty$, then the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

(iii) *Poincaré inequality: There exists a constant $C > 0$, such that:*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

(vi) *Sobolev-Poincaré inequality : there exists an other constant $C > 0$, such that:*

$$\|u\|_{L^{p*(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

The symbol $\rightharpoonup$ will denote the weak convergence, and the constants C_i, $i = 1,2,\ldots$ used in each step of proof are independent.

3 Approximate problems and A priori estimates

In this section we will prove the existence result to the approximate problems. Also we will give a uniform estimate for this solutions u_n.

Approximate problems

For $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ is defined by:

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{4}$$

Let $\Omega_n = \Omega \cap B_n(0)$ where $B_n(0)$ is the Ball with center 0 and radius n, we consider the approximate problem:

$$-\Delta_{p(.)}(u_n) + c(x,u_n) = H_n(x,u_n,\nabla u_n) + f_n - \operatorname{div} g_n \quad \text{in } \Omega_n, \\ u_n \in W_0^{1,p(.)}(\Omega_n) \cap L^\infty(\Omega_n), \tag{5}$$

with $c(x,u) = \alpha_0|u|^{p(x)-2}u$, $H_n(x,s,\xi) = T_n(H(x,s,\xi))$, $H(x,s,\xi) = d(x)\frac{|\xi|^{p(x)}}{|s|^{p(x)}+1}$, $f_n(x) = T_n(f(x))$ and $g_n(x) = \frac{g(x)}{1+\frac{1}{n}|g(x)|}$. Let us remark that $|H_n| \leq |H|$, $|H_n| \leq n$,$|f_n| \leq |f|$and $|g_n| \leq |g|$.

Lemma 1 ([13]) *Let p be a measurable function and $s > 0$ such that $sp_- > 1$ then $\||f|^s\|_{L^{p(x)}(\Omega)} = \|f\|^s_{L^{p(x)}(\Omega)}$ for every f in $L^{p(x)}(\Omega)$.*

Lemma 2 ([3]) *Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) and let us Assume that:*

$$|a(x,s,\xi)| \leq \beta(K(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{6}$$

$$a(x,s,\xi)\xi \geq \alpha|\xi|^{p(x)}, \tag{7}$$

$$[a(x,s,\xi) - a(x,s,\overline{\xi})](\xi - \overline{\xi}) > 0 \quad \text{for all } \xi \neq \overline{\xi} \text{ in } \mathbb{R}^N, \tag{8}$$

hold, and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)dx \to 0, \tag{9}$$

then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

We define the operator:
$R_n : W_0^{1,p(x)}(\Omega_n) \to W^{-1,,p'(x)}(\Omega_n)$, by:

$$\langle R_n u, v\rangle = \int_{\Omega_n} c(x,u)v - H_n(x,u,\nabla u)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega_n).$$

by the Hölder inequality we have that:
for all $u,v \in W_0^{1,p(x)}(\Omega_n)$,

$$\begin{aligned} &\Big|\int_{\Omega_n} c(x,u)v - H_n(x,u,\nabla u)v dx\Big| \\ &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\|c(x,u)\|_{L^{p'(x)}(\Omega_n)}\|v\|_{L^{p(x)}(\Omega_n)} \\ &+ \|H_n(x,u,\nabla u)\|_{L^{p'(x)}(\Omega_n)}\|v\|_{L^{p(x)}(\Omega_n)}\Big] \\ &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\Big(\int_{\Omega_n}\Big(|c(x,u)|^{p'(x)}dx + 1\Big)^{\frac{1}{p'_-}} \\ &+ \Big(\int_{\Omega_n}\Big(|H_n(x,u,\nabla u)|^{p'(x)}dx + 1\Big)^{\frac{1}{p'_-}}\Big]\|v\|_{W^{1,p(x)}(\Omega_n)} \end{aligned}$$

Then:

$$\begin{aligned} &\Big|\int_{\Omega_n} c(x,u)v + H_n(x,u,\nabla u)v dx\Big| \\ &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\Big(\int_{\Omega_n}|u|^{p(x)}\Big)dx + 1\Big)^{\frac{1}{p'_-}} \\ &+ \Big(\int_{\Omega_n} n^{p'(x)}dx + 1\Big)^{\frac{1}{p'_-}}\Big]\|v\|_{W^{1,p(x)}(\Omega_n)} \\ &\leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\Big(\int_{\Omega_n}|u|^{p(x)}\Big)dx + 1\Big)^{\frac{1}{p'_-}} \\ &+ \Big(n^{p'_+}.\operatorname{meas}(\Omega_n) + 1\Big)^{\frac{1}{p'_-}}]]\|v\|_{W^{1,p(x)}(\Omega_n)} \\ &\leq C_1\|v\|_{W^{1,p(x)}(\Omega_n)}, \end{aligned} \tag{10}$$

Lemma 3 *The operator $B_n = A + R_n$ is pseudo-monotone from $W_0^{1,p(x)}(\Omega_n)$ into $W^{-1,p'(x)}(\Omega_n)$. Moreover, B_n is coercive in the following sense*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad \text{as} \quad \|v\|_{W^{1,p(x)}(\Omega_n)} \to +\infty$$
$$\text{for} \quad v \in W_0^{1,p(x)}(\Omega_n).$$

where $Au = -\Delta_{p(x)}(u)$

Proof: Using Hölder's inequality we can show that the operator A is bounded, and by using (10) we conclude that B_n is bounded. For the coercivity, we have for any $u \in W_0^{1,p(x)}(\Omega_n)$,

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle Au, u\rangle + \langle R_n u, u\rangle \\
&= \int_{\Omega_n} |\nabla u|^{p(x)} dx + \int_{\Omega_n} c(x,u)u - H_n(x,u,\nabla u)u dx \\
&\geq \int_{\Omega} |\nabla u|^{p(x)} dx - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \quad (\text{using}(10)) \\
&\geq \|\nabla u\|^{\delta'}_{L^{p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \\
&\geq \alpha'\|u\|^{\delta'}_{W^{1,p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)}
\end{aligned}$$

(using Poincaré's inequality)

With

$$\delta' = \begin{cases} p_- & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} > 1, \\ p_+ & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} \leq 1, \end{cases}$$

Then, we obtain:

$$\frac{\langle B_n u, u\rangle}{\|u\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad \text{as } \|u\|_{W^{1,p(x)}(\Omega_n)} \to +\infty.$$

It remains now to show that B_n is pseudo-monotone. Let $(u_k)_k$ a sequence in $W_0^{1,p(x)}(\Omega_n)$ such that:

$$\begin{aligned}
&u_k \rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega_n), \\
&B_n u_k \rightharpoonup \chi \quad \text{in } W^{-1,p'(x)}(\Omega_n), \\
&\limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle.
\end{aligned} \tag{11}$$

We will prove that:

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega_n) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega_n)$, then $u_k \to u$ in $L^{p(x)}(\Omega_n)$ for a subsequence still denoted by $(u_k)_k$.

We have $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega_n)$, then $(|\nabla u_k|^{p(x)-2}\nabla u_k)_k$ is bounded in $(L^{p'(x)}(\Omega_n))^N$, therefore, there exists a function $\varphi \in (L^{p'(x)}(\Omega_n))^N$ such that:

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup \varphi \quad \text{in } (L^{p'(x)}(\Omega_n))^N \text{ as } k \to \infty. \tag{12}$$

Similarly, since $(c(x,u_k) - H_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega_n)$, then there exists a function $\psi_n \in L^{p'(x)}(\Omega_n)$ such that:

$$c(x,u_k) - H_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \quad \text{in } L^{p'(x)}(\Omega_n) \text{ as } k \to \infty, \tag{13}$$

For all $v \in W_0^{1,p(x)}(\Omega_n)$, we have:

$$\begin{aligned}
\langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)-2}\nabla u_k \nabla v dx \\
&\quad + \lim_{k\to\infty} \int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) v dx \\
&= \int_{\Omega_n} \varphi \nabla v dx + \int_{\Omega_n} \psi_n v dx.
\end{aligned} \tag{14}$$

Using (11) and (14), we obtain:

$$\begin{aligned}
&\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\
&= \limsup_{k\to\infty} \left\{ \int_{\Omega_n} |\nabla u_k|^{p(x)} dx + \int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) u_k dx \right\} \\
&\leq \int_{\Omega} \varphi \nabla u dx + \int_{\Omega} \psi_n u dx,
\end{aligned} \tag{15}$$

Thanks to (13), we have:

$$\int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) u_k dx \to \int_{\Omega_n} \psi_n u dx; \tag{16}$$

Therefore,

$$\limsup_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx \leq \int_{\Omega_n} \varphi \nabla u dx. \tag{17}$$

On the other hand, we have:

$$\int_{\Omega_n} (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_k - \nabla u) dx \geq 0, \tag{18}$$

Then

$$\int_{\Omega_n} |\nabla u_k|^{p(x)} dx \geq -\int_{\Omega_n} |\nabla u|^{p(x)} dx + \int_{\Omega_n} |\nabla u_k|^{p(x)-2}\nabla u_k \nabla u dx + \int_{\Omega_n} |\nabla u|^{p(x)-2}\nabla u \nabla u_k dx,$$

and by (12), we get:

$$\liminf_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx \geq \int_{\Omega_n} \varphi \nabla u dx,$$

this implies, thanks to (17), that:

$$\lim_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx = \int_{\Omega_n} \varphi \nabla u dx. \tag{19}$$

By combining (14), (16) and (19), we deduce that:

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Now, by (19) we can obtain:

$$\lim_{k\to+\infty} \int_{\Omega_n} (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u))(\nabla u_k - \nabla u) dx = 0,$$

In view of the Lemma 2, we obtain:

$$u_k \to u, \quad W_0^{1,p(x)}(\Omega_n), \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega_n,$$

then

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup |\nabla u|^{p(x)-2}\nabla u \qquad \text{in } (L^{p'(x)}(\Omega_n))^N,$$

and

$$c(x,u_k) - H_n(x,u_k,\nabla u_k) \rightharpoonup c(x,u) - H_n(x,u,\nabla u) \quad \text{in } L^{p'(x)}(\Omega_n),$$

we deduce that $\chi = B_n u$, which completes the proof.

By Lemma 3, we deduce that there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega_n)$ of the problem (5), (cf. [14]).

A priori estimates

Proposition 4 *Assuming that $p(.) \in C_+(\overline{\Omega})$ holds, and let u_n be any solution of (5). Then for every $\lambda > 0$ there exists a positive constant $C = C(N, p, \alpha_0, d, f, g, \lambda)$ such that:*

$$\|e^{\lambda|u_n|} - 1\|_{W_0^{1,p(x)}(\Omega_n)} \le C. \quad (20)$$

Remark 1 *The previous estimate yields an estimate for the functions $e^{\lambda|u_n|}$ in $L^{r(x)}_{loc}(\Omega)$ for every $r \in [1, +\infty)$, every $\lambda > 0$ and every set $\Omega_0 \subset\subset \Omega$, one has*

$$\|e^{|u_n|}\|_{L^{r(x)}(\Omega_0)} \le C(r_\mp, \lambda, \Omega_0)$$

Proof:

For simplicity of notation we will always omit the index n of the sequence. We take $\varphi(G_k(u))$ as test function in (5), where

$$G_k(s) = s - T_k(s) = \begin{cases} s-k & \text{if } s > k, \\ 0 & \text{if } |s| \le k, \\ s+k & \text{if } s < -k. \end{cases} \quad (21)$$

$$\text{and} \quad \varphi(s) = \left(e^{\lambda|s|} - 1\right) sign(s).$$

we have:

$$\begin{aligned} &\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \alpha_0 \int_\Omega |u|^{p(x)-1} |\varphi(G_k(u))|, \\ &\le d \int_\Omega |\nabla G_k(u)|^{p(x)} |\varphi(G_k(u))| + \int_\Omega |f||\varphi(G_k(u)) \\ &+ \int_\Omega |g||\nabla G_k(u)| \varphi'(G_k(u)) \\ &= I + J + K, \end{aligned} \quad (22)$$

For every s in $\mathbb{R}$ and if λ satisfies:

$$\lambda \ge 8d \quad (23)$$

we have:

$$d|\varphi(s)| \le \frac{1}{8} \varphi'(s) \quad (24)$$

then

$$I \le \frac{1}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) \quad (25)$$

Before estimating J, we remark that:

$$\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) = \int_\Omega |\nabla \Psi(G_k(u))|^{p(x)} \quad (26)$$

where

$$\Psi(s) = \int_0^{|s|} (\varphi'(t))^{\frac{1}{p(x)}} dt = \frac{p(x)}{\lambda^{\frac{1}{p'(x)}}} \left(e^{\frac{\lambda|s|}{p(x)}} - 1\right) \quad (27)$$

Moreover, we observe that there exists a positive constant $c_2 = c_2(p, \lambda)$ such that

$$|\varphi(s)| \le c_2 (\Psi(s))^{p(x)} \text{for every s such that} |s| \ge 1 \quad (28)$$

Now let us observe that p is a continuous variable exponent on $\overline{\Omega}$ then there exists a constant $\delta > 0$ such that:

$$\max_{y \in B(x,\delta)\cap\Omega} \frac{N - p(y)}{Np(y)} \le \min_{y \in B(x,\delta)\cap\Omega} \frac{p(y)(N - p(y))}{Np(y)} \quad \text{for all } x \in \Omega. \quad (29)$$

while $\overline{\Omega}$ is compact then we can cover it with a finite number of balls B_i for $i = 1, ..., m$ from (29) we can deduce the pointwise estimate:

$$1 < p_{-,i} \le p_{+,i} \le \frac{p_{-,i}^2 N}{N - p_{-,i} + p_{-,i}^2} < N. \quad (30)$$

is satisfy for all $i = 1, ..., m$.
$p_{-,i}, p_{+,i}$ denote the local minimum and the local maximum of p on $\overline{B_i \cap \Omega}$ respectively

Estimation of the integral J:

Let $H \ge 1$ be a constant that we will chose later. We can estimate J by splitting it as follows:

$$\begin{aligned} J &= \sum_{i=0}^m \Big[\int_{B_i \cap \{|f|>H, |G_k(u)| \ge 1\}} |f||\varphi(G_k(u))| \Big] \\ &+ \int_{\{|f|>H, |G_k(u)|<1\}} |f||\varphi(G_k(u))| + \int_{\{|f| \le H\}} |f||\varphi(G_k(u))| \\ &= J_1 + J_2 + J_3 \end{aligned}$$

By (28) J_1, can be estimated as follows

$$J_1 \le c_2 \sum_{i=0}^m \Big[\int_{B_i \cap \{|f|>H, |G_k(u)| \ge 1\}} |f| \Psi(G_k(u))^{p(x)} \Big]$$

Let ϵ a positive constant to be chosen later. Using Young, Sobolev's embedding and Lemma 1 we have:

$$\begin{aligned} J_1 &\le C \int_{\{|f|>H\}} |f|^{\frac{\epsilon N}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^m \|\nabla \Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p_{+,i}^*} \\ &\le C \int_{\{|f|>H\}} |f|^{\frac{\epsilon N}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^m \Big[\int_{B_i} |\nabla \Psi(G_k(u))|^{p(x)} \Big]^{\frac{\epsilon p_{+,i}^*}{p_{-,i}}} \end{aligned}$$

where $p^*(x) = \frac{Np(x)}{N - p(x)}$ and $p_{+,i}^* = \frac{Np_{+,i}}{N - p_{+,i}}$ since (30) we can choose ϵ such that:

$$\frac{N - p_{-,i}}{Np_{-,i}} \le \epsilon \le \frac{p_{-,i}}{p_{+,i}^*} \quad (31)$$

Then using (31) and (26) we obtain that :

$$J_1 \le C \int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{1}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u) \quad (32)$$

Remark 2 *The cases where $\|\Psi(G_k(u))\|_{L^{p^*(x)}(\Omega)} \le 1$ or $\|\nabla \Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \le 1$ are easy to see that $J_1 \le C$ (C depend on the data of the problem)*

On the other hand

$$J_2 \leq \varphi(1)\int_{\{|f|>H\}}|f| \leq \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} \tag{33}$$

Finally, if we choose k sufficiently large such that:

$$\alpha_0 k^{p_- -1} \geq 4H \tag{34}$$

We obtain:

$$\begin{aligned} J_3 &\leq \frac{\alpha_0}{4}\int_\Omega k^{p(x)-1}|\varphi(G_k(u))| \\ &\leq \frac{\alpha_0}{4}\int_\Omega |u|^{p(x)-1}|\varphi(G_k(u))| \end{aligned} \tag{35}$$

Estimation of the integral K:

Thanks to Young's inequality, we have:

$$\begin{aligned} K \leq& \frac{1}{8}\int_\Omega |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) \\ &+ C_4\int_\Omega |g|^{p'(x)}\varphi'(G_k(u)), \\ &= K_1 + K_2 \end{aligned} \tag{36}$$

The integral K_2 can be estimated as follows:

$$\begin{aligned} K_2 &\leq C_4\int_\Omega |g|^{p'(x)}\varphi'(G_k(u)), \\ &\leq C_4\lambda e^\lambda \int_{\{|G_k(u)|<1\}}|g|^{p'(x)} \\ &+ C_4\sum_{i=0}^m\Big[\int_{B_i\cap\{|g|>1,|G_k(u)|>1\}}|g|^{p'(x)}\varphi'(G_k(u))\Big] \\ &+ C_4\int_{\{|g|\leq 1,|G_k(u)|>1\}}\varphi'(G_k(u)) \\ &= K_{2,1}+K_{2,2}+K_{2,3} \end{aligned} \tag{37}$$

Since $\varphi'(s) \leq C_5(\Psi(s))^{p(x)}$ for every s such that $|s| \geq 1$, we have:

$$K_{2,2} \leq C_6\sum_{i=0}^m\Big[\int_{B_i\cap\{|g|>1,|G_k(u)|>1\}}|g|^{p'(x)}\Psi(G_k(u))^{p(x)}\Big]$$

Let ϵ be a positive constant such that (31). Using Young, Sobolev's embedding and Lemma 1 we have:

$$\begin{aligned} K_{2,2} &\leq C_7\int_{\{|g|>1\}}|g|^{\frac{\epsilon N p'(x)}{\epsilon N+p(x)-N}} + \frac{1}{8}\sum_{i=0}^m \|\nabla\Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p^*_{+,i}} \\ &\leq C_7\int_{\{|g|>1\}}|g|^{\frac{\epsilon N p'(x)}{\epsilon N+p(x)-N}} + \frac{1}{8}\sum_{i=0}^m\Big[\int_{B_i}|\nabla\Psi(G_k(u))|^{p(x)}\Big]^{\frac{\epsilon p^*_{+,i}}{p_{-,i}}} \end{aligned}$$

Then using (31) and (26) we obtain that:

$$K_{2,2} \leq C_7\int_{\{|g|>1\}}|g|^{\frac{N}{p(x)-1}} + \frac{1}{8}\int_\Omega |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u) \tag{38}$$

The same as before in the cases where $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)} \leq 1$ or $\|\nabla\Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \leq 1$ it is easy to check that $K_{2,2} \leq C$

Finally, using inequality

$$\varphi'(s) \leq C_8|\varphi(s)|, \quad \text{for every s such that} |s| \geq 1 \tag{39}$$

and choosing $k = k(p_-, \alpha_0, \lambda)$ sufficiently large such that:

$$\alpha_0 k^{p_- -1} \geq 4C_4 \tag{40}$$

we obtain:

$$\begin{aligned} K_{2,3} &\leq \frac{\alpha_0}{4}\int_\Omega k^{p(x)-1}|\varphi(G_k(u))| \\ &\leq \frac{\alpha_0}{4}\int_\Omega |u|^{p(x)-1}|\varphi(G_k(u))| \end{aligned} \tag{41}$$

Putting all the inequalities (22), (25), (32), (33), (35), (38), (41), (37) and (36) together, we get an estimate in $W_0^{1,p(x)}(\Omega)$ for $G_k(u)$,when k is large enough:

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(G_k(u))| \\ &\leq C\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} + \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} \\ &+ C_4\lambda e^\lambda\int_\Omega |g|^{p'(x)} + C_7\int_\Omega |g|^{\frac{N}{p(x)-1}} \\ &= C_9(N, p_-, p_+, \alpha_0, f, g, \lambda) \end{aligned} \tag{42}$$

For every λ, k satisfying (23), (34), (40) and for every $H \geq 1$. We fix now λ and k such that (42) holds. As before, If we take $\varphi(T_k(u))$ as a test function in (5) we obtain:

$$\begin{aligned} &\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \alpha_0\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))|, \\ &\leq d\int_\Omega |\nabla T_k(u)|^{p(x)}|\varphi(T_k(u))| + d\varphi(k)\int_\Omega |\nabla G_k(u)|^{p(x)}| \\ &+ \int_\Omega |f||\varphi(T_k(u))| + \int_\Omega |g||\nabla T_k(u)|\varphi'(T_k(u)) \\ &= L_1 + L_2 + L_3 + L_4, \end{aligned} \tag{43}$$

Using (24), we have:

$$L_1 \leq \frac{1}{4}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u) \tag{44}$$

By (42),

$$L_2 \leq C_{10}(N, p_-, p_+, \alpha_0, f, g, \lambda) \tag{45}$$

Remark 3 *if meas(Ω) is finite or if $f \in L^1(\Omega)$ it is easy to estimate the integral L_3*

In general case, let ϵ be a positive constant to be chosen later, we write

$$\begin{aligned} L_3 &\leq \varphi(k)\int_{\{|f|>1\}}|f| + \int_{\{|f|\leq 1\}}|f||\varphi(T_k(u))| \\ &\leq \varphi(k)\int_{\{|f|>1\}}|f| + \epsilon\int_\Omega |\varphi(T_k(u))|^{p(x)} \\ &+ c(\epsilon, p'_-)\int_{\{|f|\leq 1\}}|f|^{p'(x)} \end{aligned}$$

Since

$$|\varphi(T_k(u))|^{p(x)} \le C_{11}(\lambda, p_+, p_-, k)|\varphi(T_k(u))||u|^{p(x)-1},$$

choosing ϵ such that $\epsilon C_{11} \le \frac{\alpha_0}{2}$,we have:

$$L_3 \le \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))| + C_{12}(\alpha_0, f, \lambda, p_+, p_-, k) \quad (46)$$

Finally, one has

$$L_4 \le \frac{1}{4}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u) + C_{13}(\alpha_0, \lambda, p'_-, g, p_-, k) \quad (47)$$

In conclusion, putting all the estimations ((43) - (47)) together, we get:

$$\frac{1}{2}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))| \le C_{14}(N, p_-, p_+, \alpha_0, f, g, \lambda) \quad (48)$$

In view of (42) and (48), we have:

$$\int_{\{|u|\le k\}} |\nabla u|^{p(x)} e^{\lambda|u|} \le C_{15}, \quad \int_{\{|u|>k\}} |\nabla u|^{p(x)} e^{\lambda(|u|-k)} \le C_{15}$$

For every λ, k large enough (see (23), (34) and (40)), where C_{15} depends on λ, k and the data. Since

$$\int_\Omega |\nabla u|^{p(x)} e^{\lambda|u|} = \int_{\{|u|\le k\}} |\nabla u|^{p(x)} e^{\lambda|u|} + e^{\lambda k}\int_{\{|u|>k\}} |\nabla u|^{p(x)} e^{\lambda(|u|-k)} \le C_{16}$$

If we fix the value of k (depending on λ), we obtain an estimate of $|\nabla(e^{\lambda|u|}-1)|$ in $L^{p(x)}(\Omega)$ (depending on λ). This implies, by Sobolev's embedding, that:

$$\int_\Omega (e^{\lambda|u|}-1)^{p*(x)} \le C_{17} \quad (49)$$

For every λ such that (23) , where C_{17} depends on λ and on the data of the problem. Note that (49) does not imply an estimate in $L^{p(x)}(\Omega)$ for $e^{\lambda|u|}-1$, since $meas(\Omega)$ may be infinite. To obtain such an estimate, we have to combine (48) and (49) , since, for every $k > 0$, one has the inequalities

$$\int_{\{|u|\le k\}} (e^{\lambda|u|}-1)^{p(x)} \le C_{19}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u)|,$$

$$\int_{\{|u|>k\}} (e^{\lambda|u|}-1)^{p(x)} \le C_{20}\int_\Omega (e^{\lambda|u|}-1)^{p*(x)},$$

Therefore, if $k = k(\lambda)$ is such that (48) holds, we can write

$$\int_\Omega (e^{\lambda|u|}-1)^{p(x)} = \int_{\{|u|\le k\}} (e^{\lambda|u|}-1)^{p(x)} + \int_{\{|u|>k\}} (e^{\lambda|u|}-1)^{p(x)} \le C_{21} \quad (50)$$

where C_{21} depends on λ and the data of the problem.

4 Main results

In this section we will prove the main result of this paper. Let $\{u_n\}$ be any sequence of solutions of problem (5), we extend them to zero in $\Omega \setminus \Omega_n$. By (20), there exist a subsequence (still denoted by u_n) and a function $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$.

Theorem 1 *There exists at least one solution u of (1); which is such that*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\psi dx + \int_\Omega c(x,u)\psi dx + \int_\Omega H(x,u,\nabla u)\psi dx = \int_\Omega f\psi dx - \int_\Omega g\nabla\psi dx. \quad (51)$$

for every function $\psi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Moreover u satisfies

$$e^{\lambda|u|}-1 \in W_0^{1,p(x)}(\Omega) \quad (52)$$

for every $\lambda \ge 0$.

The proof will be made in three steps.

Step 1: An estimate for $\int_\Omega |\nabla G_k(u_n)|^{p(x)}$

In view of (42) we have:

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)} \le \frac{2C}{\lambda}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2\varphi(1)}{\lambda H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2C_4\lambda e^\lambda}{\lambda}\int_\Omega |g|^{p'(x)} + \frac{2C_7}{\lambda}\int_\Omega |g|^{\frac{N}{p(x)-1}} \quad (53)$$

If η is an arbitrary positive number, let us choose H such that the right-hand side of (53) is smaller than η. It follows that, for every k satisfying (34), (40) , every λ satisfying (23), and every $n \in \mathbb{N}$

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)} \le \eta$$

which proves:

$$\sup_n \int_\Omega |\nabla G_k(u_n)|^{p(x)} \to 0 \quad \text{as} \quad k \to \infty \quad (54)$$

Step 2: Strong convergence of $\nabla T_k(u_n)$

In this step, we will fix $k > 0$ and prove that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $L^{p(x)}(\Omega_0; \mathbb{R}^N)$ as $n \to \infty$; for k fixed. In order to prove this result we define:

$$z_n(x) = T_k(u_n) - T_k(u)$$

and we choose ψ a cut-off function such that

$$\psi \in C_0^\infty(\Omega), \quad 0 \le \psi \le 1, \quad \psi = 0 \quad \text{in} \quad \Omega_0$$

Let us take:

$$v = \varphi(z_n)e^{\delta|u_n|}\psi \tag{55}$$

as a test function in (5), where λ and δ are a positive constant to be chosen later, we obtain:

$$\begin{aligned} A_n + B_n &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla z_n \varphi'(z_n)e^{\delta|u_n|}\psi \\ &+ \int_\Omega c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi \\ &\leq d\int_\Omega |\nabla u_n|^{p(x)}|\varphi(z_n)|e^{\delta|u_n|}\psi + \int_\Omega |f||\varphi(z_n)|e^{\delta|u_n|}\psi \\ &- \delta\int_\Omega |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi \\ &+ \int_\Omega |\nabla u_n|^{p(x)-1}|\nabla\psi||\varphi(z_n)|e^{\delta|u_n|} \\ &+ \int_\Omega |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\ &+ \delta\int_\Omega |g||\nabla u_n||\varphi(z_n)|e^{\delta|u_n|}\psi + \int_\Omega |g||\nabla\psi||\varphi(z_n)|e^{\delta|u_n|} \\ &= C_n + D_n + E_n + F_n + G_n + H_n + L_n \end{aligned} \tag{56}$$

Splitting Ω into $\Omega = \{|u_n| \leq k\} \cup \{|u_n| > k\}$ we can write:

$$\begin{aligned} A_n &= \int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla z_n\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\ &+ \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla z_n\varphi'(z_n)e^{\delta|u_n|}\psi \\ &= \int_{\{|u_n|\leq k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)] \\ &* \nabla z_n\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\ &+ \int_{\{|u_n|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla z_n\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\ &+ \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla z_n\varphi'(z_n)e^{\delta|u_n|}\psi \\ &= A_{1,n} + A_{2,n} + A_{3,n} \end{aligned}$$

since

$$|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}} \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\leq k\}}$$

almost everywhere in Ω (on the set where $|u(x)| = k$ we have $|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) = 0$)and

$$\begin{aligned} ||\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}}| \\ \leq |\nabla u|^{p(x)-1}\varphi'(2k)e^{\delta k}\psi \end{aligned}$$

which is a fixed function in $L^{p'(x)}(\Omega)$. Therefore by Lebesgue's theorem we have

$$|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}} \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\leq k\}}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $A_{2,n} \to 0$. Similarly, since $\nabla z_n\chi_{\{|u_n|>k\}} = -\nabla T_k(u)\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $|\nabla u_n|^{p(x)-2}\nabla u_n\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (6), (20) and Remark 1 we obtain $A_{3,n} \to 0$. Therefore, we have proved that:

$$A_n = A_{1,n} + o(1) \tag{57}$$

For the integral B_n while $\varphi(z_n)$ has the same sign as $c(u_n)$ on the set $\{|u_n| > k\}$ we have

$$\begin{aligned} B_n &= \int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi \\ &+ \int_{\{|u_n|>k\}} c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi \\ &\geq \int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi \end{aligned}$$

the last integrand converges pointwise and it is bounded then $\int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi$ goes to zero. Therefore, we obtain that:

$$B_n \geq o(1) \tag{58}$$

Let us examine C_n and D_n together. We first fix δ such that

$$\delta > d$$

Since $\varphi(z_n)sign(u_n) = |\varphi(z_n)|$ on the set $\{|u_n| > k\}$ we have

$$\begin{aligned} C_n + E_n &\leq d\int_\Omega |\nabla u_n|^{p(x)}|\varphi(z_n)|e^{\delta|u_n|}\psi \\ &- \delta\int_\Omega |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi \\ &\leq (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)}|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &+ (d-\delta)\int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}\psi \\ &\leq (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)}|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &= (d+\delta)\int_{\{|u_n|\leq k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \\ &- |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &+ (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla T_k(u)| \\ &\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\ &+ (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \end{aligned}$$

The last two integrals converge to zero.If we choose λ such that:

$$\lambda \geq 2(d+\delta)$$

we have:

$$(d+\delta)|\varphi(s)| \leq \frac{\varphi'(s)}{2} \quad \text{for every s in } \mathbb{R}$$

then we can obtain:

$$C_n + E_n \leq \frac{1}{2}A_{1,n} + o(1) \tag{59}$$

Using Remark 1 we can observe that:

$$D_n \to 0 \tag{60}$$

For the term F_n we can see that $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$. by (20) the term $|\nabla u_n|^{p(x)-2}\nabla u_n e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that:

$$F_n \to 0 \tag{61}$$

For the term G_n like before we have:

$$\begin{aligned} G_n &= \int_{\{|u|\leq k\}} |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\ &+ \int_{\{|u|>k\}} |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\ &= G_{1,n} + G_{2,n} \end{aligned}$$

since

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\leq k\}} \to |g|\varphi'(0)e^{\delta|T_k u|}\psi\chi_{\{|u|\leq k\}}$$

almost everywhere in Ω and

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\leq k\}} \leq |g|\varphi'(2k)e^{\delta k}\psi$$

Therefore by Lebesgue's theorem we have:

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\leq k\}} \to |g|\varphi'(0)e^{\delta|T_k u|}\psi\chi_{\{|u|\leq k\}}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $G_{1,n} \to 0$. Similarly, since $|\nabla z_n|\chi_{\{|u_n|>k\}} = |\nabla T_k(u)|\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $|g|\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (20) and remark 1 we obtain $G_{2,n} \to 0$. Therefore, we have proved that:

$$G_n \to 0 \tag{62}$$

Moreover

$$|g||\varphi(z_n)|\psi \to 0$$

almost everywhere in Ω and

$$|g||\varphi(z_n)|\psi \leq |g||\varphi(2k)|\psi$$

Therefore by Lebesgue's theorem we have:

$$|g||\varphi(z_n)|\psi \to 0$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla u_n e^{\delta|u_n|} \rightharpoonup \nabla u e^{\delta|u|}$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^{\mathbb{N}})$, then:

$$H_n \to 0 \tag{63}$$

Finally, $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$. by (20) the term $|g|e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that:

$$L_n \to 0 \tag{64}$$

Putting all inequalities (56), (57), (58), (59), (60), (61), (62), (63) and (64) we can conclude:

$$A_{1,n} \to 0 \tag{65}$$

On the other hand we have

$$\int_{\{|u_n|>k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla z_n$$
$$\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi = \int_{\{|u_n|>k\}} |\nabla T_k(u)|^{p(x)}\varphi'(k - T_k(u))e^{\delta k}\psi \to 0 \tag{66}$$

From (65) and (66) we can conclude that:

$$\int_{\Omega_0} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]$$
$$(\nabla T_k(u_n) - \nabla T_k(u)) \to 0 \tag{67}$$

Finally, using the Lemma 2 we have:

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{68}$$

Step 3: End of the proof

Observing that:

$$\nabla u_n - \nabla u = \nabla T_k u_n - \nabla T_k u + \nabla G_k u_n - \nabla G_k u$$

Let Ω_0 be an open set compactly contained in Ω, using (54) and (68) we have:

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{69}$$

To obtain (51) we have to pass to the limit in the distributional formulation of problem (5) using (69). Finally, statement (52) follows easily from Proposition 4 and (69), using Fatou's Lemma.

5 Boundedness of solutions

In this section we will gave some regularity on the solution of the problem (1) using an adaptation of a classical technique due to Stampacchia. To do this we need the following lemma (see [15]):

Lemma 4 *Let ϕ be a non-negative, non-increasing function defined on the halfline $[k_0,\infty)$. Suppose that there exist positive constants A, μ,β, with $\beta > 1$, such that*

$$\phi(h) \leq \frac{A}{(h-k)^\mu}\phi(k)^\beta$$

for every $h > k \geq k_0$. Then $\phi(k) = 0$ for every $k \geq k_1$, where

$$k_1 = k_0 + A^{1/\mu}2^{\beta/(\beta-1)}\phi(k_0)^{(\beta-1)/\mu}$$

The result that we are going to prove is the following:

Theorem 2 *Suppose that* (3) *holds. Then every solution u of* (1)*; which is specified in* (4) *is essentially bounded, and*

$$\|u_n\|_{L^\infty(\Omega)} \leq C \tag{70}$$

The proof relies on the combined use of the well-known technique by Stampacchia (see [15]) and suitable exponential test functions, as in [16].

Proof: Since (42) we can obtain an estimate for $\int_\Omega |u|^{p(x)-1}\varphi(G_k(u))$ then for some constant $k_0 = k(\lambda)$ sufficiently large we have

$$meas(A_{k_0}) < 1 \tag{71}$$

where

$$A_k = \{x \in \Omega : |u| > k\}$$

as before we can take the test function $\varphi(G_k(u))$ then we have:

$$\begin{aligned}&\int_{A_k} |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \alpha_0 \int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|,\\ &\leq d\int_{A_k} |\nabla G_k(u)|^{p(x)}|\varphi(G_k(u))| + \int_{A_k\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\ &+ \int_{A_k\cap\{|f|\leq 1\}} |\varphi(G_k(u))| + \int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u))\end{aligned} \tag{72}$$

As in the proof of Proposition 4 one has:

$$\begin{aligned}&\int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u))\\ &\leq \frac{1}{4}\int_{A_k} |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + C_{22}\int_{A_k} |g|^{p'(x)}\varphi'(G_k(u))\end{aligned}$$

and if $\lambda \geq 4d$ and $k \geq k_0(\lambda)$(large enough) where

$$\alpha_0 k_0^{p_- -1} \geq 4 \tag{73}$$

then

$$\begin{aligned}&\frac{1}{2}\int_{A_k} |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \frac{3\alpha_0}{4}\int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|,\\ &\leq \int_{(A_k\setminus A_{k+1})\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\ &+ \int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))| + C_{22}\varphi'(1)\int_{A_k\setminus A_{k+1}} |g|^{p'(x)}\\ &+ C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))\end{aligned} \tag{74}$$

using Hölder inequality we have:

$$\begin{aligned}&\int_{(A_k\setminus A_{k+1})\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\ &\leq \varphi(1)\left(\frac{1}{q_-} + \frac{1}{q'_-}\right)\|f\|_{L^{q(x)}(\{|f|>1\})}(meas(A_k))^{\frac{1}{q'_+}}\end{aligned}$$

by Hölder's inequality and interpolation we obtain:

$$\begin{aligned}&\int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\ &\leq \|f\|_{L^{q_-}(A_{k+1}\cap\{|f|>1\})}\|\varphi(G_k(u))\|^{\frac{N}{p_- q_-}}_{L^{p*_-/p_-}(A_{k+1})}\|\varphi(G_k(u))\|^{1-\frac{N}{p_- q_-}}_{L^1(A_{k+1})}\end{aligned}$$

while (28), (71) and using Young's and sobolev's inequalities we can deduce that:

$$\begin{aligned}&\int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\ &\leq \frac{1}{8}\|\nabla\psi(G_k(u))\|^{p_-}_{L^{p(x)}(A_k)}\\ &+ C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}\\ &\leq \frac{1}{8}\int_{A_k} |\nabla\psi(G_k(u))|^{p(x)} + 1\,dx\\ &+ C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}\\ &\leq \frac{1}{8}\int_{A_k} |\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx + \frac{1}{8}meas(A_k)\\ &+ C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}\end{aligned}$$

Therfore, choosing k_0 such that:

$$C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})} \leq \frac{\alpha_0 k_0^{p_- -1}}{4} \tag{75}$$

the second integral in the right-hand side of (74) can be absorbed by the left-hand side.
In view of Hölder's inequality and (71) and (3) we have:

$$\begin{aligned}&C_{22}\varphi'(1)\int_{A_k\setminus A_{k+1}} |g|^{p'(x)}\\ &\leq C_{23}\left(\int_{A_k} |g|^{p'(x)}\right)^{\eta}(meas(A_k))^{1-\frac{p'_+}{r_-}}\\ &\leq C_{24}(\|g\|_{L^{r(x)}(A_k)})^{\delta''}(meas(A_k))^{1-\frac{p'_+}{r_-}}\end{aligned}$$

where

$$\eta = \begin{cases}\frac{p'_-}{r_+} & \text{if } \int_{A_k} |g|^{p'(x)} \leq 1,\\ \frac{p'_+}{r_-} & \text{if } \int_{A_k} |g|^{p'(x)} > 1.\end{cases}$$

$$\delta'' = \begin{cases}\frac{\eta}{r_-} & \text{if } \|g\|_{L^{r(x)}(A_k)} \geq 1,\\ \frac{\eta}{r_+} & \text{if } \|g\|_{L^{r(x)}(A_k)} < 1.\end{cases}$$

Finally, with similar calculations, using (39) we have:

$$\begin{aligned}&C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))\\ &\leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}} |g|^{p'(x)}|\varphi(G_k(u))|\\ &+ C_{24}\int_{A_{k+1}\cap\{|g|\leq 1\}} |\varphi(G_k(u))|\end{aligned}$$

If we choose k_0 such that:

$$\alpha_0 k_0^{p_- -1} > 4C_{24} \tag{76}$$

then

$$\begin{aligned}&C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))\\ &\leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}} |g|^{p'_+}|\varphi(G_k(u))|\\ &+ \frac{\alpha_0}{4}\int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|\end{aligned}$$

by Hölder's inequality and interpolation we obtain:

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$

$$\leq C_{24}\|g\|^{p'_+}_{L^{r_-}(\Omega,\mathbb{R}^N)}\|\varphi(G_k(u))\|^{\frac{p'_+N}{p_-r_-}}_{L^{p*_-/p_-}(A_k)}\|\varphi(G_k(u))\|^{1-\frac{p'_+N}{p_-r_-}}_{L^1(A_k)}$$

$$+\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

as before while (28), (71) and using Young's and sobolev's inequalities we can deduce that:

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$
$$\leq \frac{1}{8}\int_{A_k}|\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx+\frac{1}{8}meas(A_k)$$
$$+C_{25}\|g\|^{\frac{p'_+p_-r_-}{p_-r_--p'_+N}}_{L^{r_-}(\Omega,\mathbb{R}^N)}\|\varphi(G_k(u))\|_{L^1(A_k)}$$
$$+\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

Therefore, by taking k_0 satistying (71), (73), (75), (76) and the further condition:

$$C_{25}\|g\|^{\frac{p'_+p_-r_-}{p_-r_--p'_+N}}_{L^{r_-}(\Omega,\mathbb{R}^N)}\leq\frac{\alpha_0k_0^{p_--1}}{4} \tag{77}$$

one obtains, for every $k\geq k_0$:

$$\frac{1}{4}\int_{A_k}|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u))$$
$$\leq\varphi(1)(\frac{1}{q_-}+\frac{1}{q'_-})\|f\|_{L^{q(x)}(\{|f|>1\})}(meas(A_k))^{\frac{1}{q'_+}}+\frac{1}{4}meas(A_k)$$
$$+C_{24}(\|g\|_{L^{r(x)}(A_k)})^{\delta''}(meas(A_k))^{1-\frac{p'_+}{r_-}}$$
$$\leq C_{26}(meas(A_k))^m$$

where $m=\min(\frac{1}{q'_+},1-\frac{p'_+}{r_-})$ in view of (26) and Sobolev's inequality we can obtain:

$$(\int_{A_k}|\psi(G_k(u))|^{p*(x)})^\beta\leq\|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}$$
$$\leq C_{27}(meas(A_k))^{\frac{m}{\alpha}}$$

Where:

$$\alpha=\begin{cases}p_+ & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)}\leq 1,\\ p_- & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)}>1.\end{cases}$$

$$\beta=\begin{cases}\frac{N-p_-}{Np_-} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}\leq 1,\\ \frac{N-p_+}{Np_+} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}>1.\end{cases}$$

We now take $h-k>1$ and recall that there exists $C_{28}(\lambda,p_+,p_-)$such that $|\psi(s)|\geq C_{28}|s|$ for every $s\in\mathbb{R}$ so that

$$[C_{28}(h-k)]^{p*_-}meas(A_h)\leq\int_{A_h}|\psi(G_k(u))|^{p*(x)}$$
$$\leq\int_{A_k}|\psi(G_k(u))|^{p*(x)}$$
$$\leq C_{29}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Then it follows for every h and k (such that $h>k\geq k_0$) that

$$meas(A_h)\leq\frac{C_{30}}{[h-k]^{p*_-}}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Since by (3), $\frac{m}{\alpha\beta}>1$ Lemma 4 applied to the function $\phi(h)=meas(A_h)$ gives:

$$\|u_n\|_{L^\infty(\Omega)}\leq C$$

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