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On the Spectrum of problems involving both p(x)-Laplacian and P(x)-Biharmonic

Abdelouahed El Khalil[1], My Driss Morchid Alaoui[*,2], Abdelfattah Touzani[2]

[1] *Department of Mathematics and Statistics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 90950, 11623 Riyadh, KSA*

[2] *Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, P.O. Box 1796 Atlas Fez, Morocco*

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ABSTRACT

We prove the existence of at least one non-decreasing sequence of positive eigenvalues for the problem

$$\begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u = \lambda|u|^{p(x)-2}u, & in\ \Omega \\ u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \end{cases}$$

Our analysis mainly relies on variational arguments involving Ljusternik-Schnirelmann theory.

1 Introduction

Consider the following nonlinear eigenvalue problem

$$\begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u = \lambda|u|^{p(x)-2}u, & \text{in } \Omega \\ u \in W_0^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 4)$.
The real λ is a parameter which plays the role of eigenvalue. For a function $p(.) \in C(\overline{\Omega})$, we assume the following hypothesis

$$1 < p^- = \min_{x\in\overline{\Omega}} p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < +\infty. \tag{1.2}$$

$\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator which is a natural generalization of the p-biharmonic (where the exponent p is constant) and $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-harmonic operator.

It is well known that elliptic equations involving the non-standard growth are not trivial generalizations of similar problems studied in the constant case since the non-standard growth operator is not homogeneous and, thus, some techniques which can be applied in the case of the constant growth operators will fail in this new situation, such as the Lagrange multiplier theorem, see, e.g [1, 2]. Problerms with $p(x)$-growth conditions are an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics.

Recently for the case $p(x) \equiv p$ constant Giri, Choudhuri and Pradhan [3] proved the existence and concentration phenomena of solutions on the set $V^{-1}\{0\}$ for the following p-biharmonic elliptic equation:
$\Delta^2_p u - \Delta_p u + \lambda V(x)|u|^{p-2}u = f(x,u) \quad x \in \mathbb{R}^{\mathbb{N}}$, as $\lambda \to \infty$ unther some assumptions on the nonlinear function f. By variational methods, Lihua Liu and Caisheng Chen [4] establish the existence of infinitely many high-energy solutions to the equation

$$\Delta^2_p u - \Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^{\mathbb{N}},$$

with a concave-convex nonlinearity,i.e.,
$f(x,u) = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$, $1 < m < p < q < p^* = \frac{pN}{N-2p}$.
In our case by using the Ljusternik-Schnirelmann theory we obtain the existence of infinitely many solutions for the problem (1.1).

The outline of the rest of the paper is as follows. In Section 2 we present some definitions and basic resuls that are necessary. In Section 3 we give the proof of our main result about existence of solutions for problem (1.1).

[*]Corresponding Author: My Driss Morchid Alaoui, FSDM , Fez, Morocco& morchid_driss@yahoo.fr

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2 Preliminaries and Useful results

We state some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$. We refer the reader to the monograph by [5] and to the references therein. Define the generalized Lebesgue space by

$$L^{p(.)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\},$$

endowed with the Luxemburg norm

$$|u|_{p(.)} = \inf\Big\{\mu > 0 : \int_\Omega \Big|\frac{u}{\mu}\Big|^{p(x)} dx \le 1\Big\}$$

To manipulate this spaces better, we use the modular mapping

$$\rho : L^{p(.)}(\Omega) \to \mathbb{R}$$

defined by

$$\rho(u) = \int_\Omega |u|^{p(x)}\, dx$$

Proposition 2.1 (6) *Under the hypothesis* (1.2), *the space* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *is separable, uniformly convex, reflexive and its conjugate dual space is* $L^{p'(.)}(\Omega)$ *where* $p'(.)$ *is the conjugate function of* $p(.)$, *related by*

$$p'(x) = \frac{p(x)}{p(x)-1}, \quad \forall x \in \Omega.$$

For $u \in L^{p(.)}(\Omega)$ *and* $v \in L^{p'(.)}(\Omega)$ *we have*

$$\Big|\int_\Omega u(x)v(x)\, dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(.)}|v|_{p'(.)} \le 2|u|_{p(.)}|v|_{p'(.)}.$$

Sobolev space with variable exponent $W^{m,p(.)}(\Omega)$ are defined as

$$W^{m,p(.)}(\Omega) = \Big\{u \in L^{p(.)}(\Omega) : D^\alpha u \in L^{p(.)}(\Omega), |\alpha| \le m\Big\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$, (the derivation in distributions sense) with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{m,p(.)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(x)} = \sum_{|\alpha| \le m} |D^\alpha u|_{p(x)},$$

is a Banach, separable and reflexive space. For more details, we refer the reader to [6, 7, 8] and [9]. We denote by $W_0^{m,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$.
Note that the weak solutions of the problem (1.1) are considered in the Sobolev space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ is equiped with the norm

$$\|u\|_{p(x)} = |\triangle u|_{p(x)} + |\nabla u|_{p(x)}$$

In the sequel, we Set

$$X = W_0^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

Then, endowed with the norm $\|u\|_{p(x)}$, X is a separable and reflexive Banach space. Moreover, $\|.\|_{p(x)}$ and $|\triangle u|_{p(x)}$ are two equivalent norms of X by [10, Theorem4.4].
Let

$$\|u\| = \inf\{\mu > 0; \int_\Omega \Big(\Big|\frac{\Delta u}{\mu}\Big|^{p(x)} + \Big|\frac{\nabla u}{\mu}\Big|^{p(x)}\Big) dx \le 1\},$$

Then, $\|u\|$ is equivalent to the norms $\|.\|_{p(x)}$ and $|\triangle u|_{p(x)}$ in X.

Lemma 2.2 (6) *For all* $p, r \in C_+(\overline{\Omega})$ *such that* $r(x) \le p_m^*(x)$ *for all* $x \in \overline{\Omega}$, *then there is a continuous and compact embedding* $W^{m,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, *where*

$$p_m^*(x) = \begin{cases} \frac{Np(x)}{N-mp(x)}, & \text{if } mp(x) < N; \\ +\infty, & \text{if } mp(x) \ge N. \end{cases}$$

Proposition 2.3 *Let* $I(u) = \int_\Omega (|\frac{\Delta u}{\mu}|^{p(x)} + |\frac{\nabla u}{\mu}|^{p(x)}) dx$, *for* $u \in L^{p(.)}$, *we have*

(1) $\|u\| < (=; > 1) \Leftrightarrow I(u) < (=; > 1)$

(2) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}$

(3) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}$

(4) $\|u\| \to 0(resp \to +\infty) \Leftrightarrow I(u) \to 0, (resp \to +\infty)$

The proof of this proposition is similar to the proof of [6, Theorem 1.3] .

Recall that our main result of this work is to show that problem (1.1) has at least one non-decreasing sequence of nonnegative eigenvalues $(\lambda_k)_{k\ge1}$. To attain this objective we will use a variational technique based on Ljusternick-Schnirelmann theory on C^1-manifolds [11]. In fact, we give a direct characterization of λ_k involving a mini-max argument over sets of genus greater than k.

We set

$$\lambda_1 = \inf\Big\{\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\, dx, u \in X, \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx = 1\Big\} \tag{2.1}$$

The value defined in (2.1) can be written as the Rayleigh quotient

$$\lambda_1 = \inf \frac{\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\, dx,}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)dx}}, \tag{2.2}$$

where the infimum is taken over $X \setminus \{0\}$.

Definition 2.4 *Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle .,. \rangle$. We shall deal with mappings T acting from X into X^*. The strong convergence in X (and in X^*) is denoted by $\to$ and the weak convergence by $\rightharpoonup$. T is said to belong to the class (S^+), if for any sequence u_n in X converging weakly to $u \in X$ and $\limsup_{n\to+\infty} \langle T, u_n - u \rangle \leq 0$, it follows that u_n converges strongly to u in X. We write $T \in (S^+)$.*

Consider the following two functionals defined on X:

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} (|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\, dx \text{ and } \varphi(u) = \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx,$$

and set $\mathcal{M} = \{u \in X; \varphi(u) = 1\}$.

Lemma 2.5 *We have the following statements*

(i) Φ and φ are even, and of class C^1 on X.

(ii) $\mathcal{M}$ is a closed C^1-manifold.

Proof. It is clear that φ and Φ are even and of class C^1 on X and $\mathcal{M} = \varphi^{-1}\{1\}$. Therefore $\mathcal{M}$ is closed. The derivative operator φ' satisfies $\varphi'(u) \neq 0\ \forall u \in \mathcal{M}$ (i.e., $\varphi'(u)$ is onto for all $u \in \mathcal{M}$). Hence φ is a submersion, which proves that $\mathcal{M}$ is a C^1-manifold. Let as split Φ on two functionals.

$$\Phi(u) = \Phi_1(u) + \Phi_2(u),$$

where

$$\Phi_1 = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}\, dx; \quad \Phi_2 = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx.$$

Now we consider the operator $T_1 := \Phi_1' : W_0^{2,p(.)}(\Omega) \to W^{-2,p'(.)}(\Omega)$ is defined as

$$\langle T_1(u), v \rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v\, dx, \quad \text{for any } u, v \in W_0^{2,p(.)}(\Omega),$$

and the p(x)-laplace operator $-\Delta_{p(x)} := T2 := \Phi_2' : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega)$ as

$$\langle -\Delta_{p(x)}(u), v \rangle = \langle T_2(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx, \quad \text{for } u, v \in W_0^{1,p(.)}(\Omega),$$

Lemma 2.6 *The following statements hold*

(i) T_1 is continuous, bounded and strictly monotone.

(ii) T_1 is of (S_+) type.

(iii) T_2 is a homeomorphism.

Proof.

(i) We recall the following well-known inequalities, which hold for any three real a, b and p

$$\Big(a|a|^{p-2} - b|b|^{p-2}\Big)(a-b) \geq c(p) \begin{cases} |a-b|^p, & \text{if } p \geq 2 \\ \dfrac{|a-b|^2}{(|a|+|b|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases} \tag{2.3}$$

where $c(p) = 2^{2-p}$ when $p \geq 2$ and $c(p) = p - 1$ when $1 < p < 2$.

Let $(u_n)_n \subset W_0^{2,p(.)}(\Omega)$ and $u_n \rightharpoonup u$ (weakly) in $W_0^{2,p(.)}(\Omega)$. Therefore we have for $p(\cdot) \geq 2$.

$$\begin{aligned} 2^{2-p^+} &\int_{\{x\in\Omega:\, p(x)\geq 2\}} |\Delta u_n - \Delta u|^{p(x)}\, dx \\ &\leq \int_{\{x\in\Omega:\, p(x)\geq 2\}} \Big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\Big) \Big(\Delta u_n - \Delta u\Big) dx \\ &\leq \int_\Omega \Big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\Big)\Big(\Delta u_n - \Delta u\Big) dx \\ &:= \varepsilon(\tfrac{1}{n}). \end{aligned} \tag{2.4}$$

On the set where $1 < p(\cdot) < 2$, we employ (2.3) as follows:

$$\begin{aligned} &\int_{\{x\in\Omega:\, 1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}\, dx \\ &\leq \int_{\{x\in\Omega:\, 1<p(x)<2\}} \frac{|\Delta u_n - \Delta u|^{p(x)}}{(|\Delta u_n| + |\Delta u|)^{\frac{p(x)(2-p(x))}{2}}} (|\Delta u_n| + |\Delta u|)^{\frac{p(x)(2-p(x))}{2}}\, dx \\ &\leq 2 \left\| \frac{|\Delta u_n - \Delta u|^{p(x)}}{(|\Delta u_n|+|\Delta u|)^{\frac{p(x)(2-p(x))}{2}}} \right\|_{L^{2/p(x)}(\Omega)} \times \left\| (|\Delta u_n| + |\Delta u|)^{\frac{p(x)(2-p(x))}{2}} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega)} \\ &\leq 2 \max\left\{ \left(\int_\Omega \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p(x)}}\, dx \right)^{\frac{p^-}{2}}, \left(\int_\Omega \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p(x)}}\, dx \right)^{\frac{p^+}{2}} \right\} \\ &\quad \times \max\left\{ \left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\, dx \right)^{\frac{2-p^-}{2}}; \left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\, dx \right)^{\frac{2-p^+}{2}} \right\} \\ &\leq 2 \max\left\{ \Big(p^- - 1\Big)^{\frac{-p^-}{2}} \Big(\varepsilon(\frac{1}{n})\Big)^{\frac{p^-}{2}}; \Big(p^- - 1\Big)^{\frac{-p^+}{2}} \Big(\varepsilon(\frac{1}{n})\Big)^{\frac{p^+}{2}} \right\} \\ &\quad \times \max\left\{ \left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\, dx \right)^{\frac{2-p^-}{2}}; \left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\, dx \right)^{\frac{2-p^+}{2}} \right\}. \end{aligned} \tag{2.5}$$

Since u_n is bounded in X, implies that $\varepsilon(\frac{1}{n}) \to 0$ as $n \to \infty$. Hence, sending n to ∞ in (2.4) and (2.5), we obtain

$$\lim_{n\to\infty} \int_\Omega |\Delta u_n - \Delta u|^{p(x)}\, dx = 0.$$

Since T_1 is the Fréchet derivative of Φ_1, it follows that T_1 is continuous and bounded so that we deduce that for all $u, v \in W_0^{2,p(.)}(\Omega)$ such that $u \neq v$,

$$\langle T_1(u) - T_1(v), u - v \rangle > 0.$$

This means that T_1 is strictly monotone.

(ii) Let $(u_n)_n$ be a sequence of X such that $u_n \rightharpoonup u$ weakly in $W_0^{2,p(.)}(\Omega)$ and $\limsup_{n\to+\infty}\langle T_1(u_n), u_n - u\rangle \le 0$. From (2.3), we have

$$\langle T_1(u_n) - T_1(u), u_n - u\rangle \ge 0, \qquad (2.6)$$

and since $u_n \rightharpoonup u$ weakly in $W_0^{2,p(.)}(\Omega)$, it follows that

$$\limsup_{n\to+\infty}\langle T_1(u_n) - T_1(u), u_n - u\rangle = 0. \qquad (2.7)$$

Thus again from (2.3), we have

$$\begin{cases} \int_{\{x\in\Omega: p(x)\ge 2\}} |\Delta u_n - \Delta u|^{p(x)}\,dx \\ \le 2^{(p^- -2)}\int_\Omega A(u_n,u)\,dx, \\ \int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \\ \le (p^+ - 1)\int_\Omega (A(u_n,u))^{\frac{p(x)}{2}}(B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx, \end{cases}$$

where

$$\begin{cases} A(u_n,u) = \\ (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n - \Delta u), \\ B(u_n,u) = (|\Delta u_n| + |\Delta u|)^{2-p(x)}. \end{cases}$$

On the other hand, by (2.6) and since

$$\int_\Omega A(u_n,u)\,dx = \langle T_1(u_n) - T_1(u), u_n - u\rangle,$$

we can consider $0 \le \int_\Omega A(u_n,u)\,dx < 1$.
We distinguish two cases:
First, If $\int_\Omega A(u_n,u)\,dx = 0$, then $A(u_n,u) = 0$, since $A(u_n,u) \ge 0$ a.e. in Ω.
Second, If $0 < \int_\Omega A(u_n,u)\,dx < 1$. Thus

$$t^{p(x)} := \left(\int_{\{x\in\Omega:1<p(x)<2\}} A(u_n,u)\,dx\right)^{-1} \text{ is positive}$$

and by applying Young's inequality we deduce that

$$\int_{\{x\in\Omega:1<p(x)<2\}} \left[t(A(u_n,u))^{\frac{p(x)}{2}}\right](B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx$$
$$\le \int_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)(t)^{\frac{2}{p(x)}} + (B(u_n,u))^{p(x)}\right)dx$$

The fact that $\frac{2}{p(x)} < 2$, we have

$$\int_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)(t)^{\frac{2}{p(x)}} + (B(u_n,u))^{p(x)}\right)dx$$
$$\le \int_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)t^2 + (B(u_n,u))^{p(x)}\right)dx$$
$$\le 1 + \int_{\{x\in\Omega:1<p(x)<2\}} (B(u_n,u))^{p(x)}\,dx.$$

Hence.

$$\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx$$
$$\le \left(\int_{\{x\in\Omega:1<p(x)<2\}} A(u_n,u)\,dx\right)^{\frac{1}{2}}\left(1 + \int_\Omega (B(u_n,u))^{p(x)}\,dx\right).$$

Since $\int_\Omega (B(u_n,u))^{p(x)}\,dx$ is bounded, then

$$\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \to 0 \text{ as } n\to\infty$$

(iii) Note that the strict monotonicity of T_1 implies that T_1 is into operator.

Moreover, T_1 is a coercive operator. Indeed, from Proposition 2.3 and since $p^- - 1 > 0$, for each $u \in W_0^{2,p(x)}(\Omega)$ such that $\|u\| \ge 1$, we have

$$\frac{\langle T_1(u), u\rangle}{\|u\|} = \frac{\Phi'(u)}{\|u\|} \ge \|u\|^{p^- -1} \to \infty, \quad \text{as } \|u\| \to \infty.$$

Finally, thanks to Minty-Browder Theorem [12], the operator T_1 is an surjection and admits an inverse mapping.
To complete the proof of (iii), it suffices then to show the continuity of T_1^{-1}. Indeed, let $(f_n)_n$ be a sequence of $W^{-2,p'(.)}(\Omega)$ such that $f_n \to f$ in $W^{-2,p'(.)}(\Omega)$. Let u_n and u in $W_0^{2,p(.)}(\Omega)$ such that Since $\int_\Omega (B(u_n,u))^{p(x)}\,dx$ is bounded, then $\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \to 0$ as $n\to\infty$

(iii) Note that the strict monotonicity of T_1 implies that T_1 is into operator.
Moreover, T_1 is a coercive operator. Indeed, from Proposition 2.3 and since $p^- - 1 > 0$, for each $u \in W_0^{2,p(x)}(\Omega)$ such that $\|u\| \ge 1$, we have $\frac{\langle T_1(u),u\rangle}{\|u\|} = \frac{\Phi'(u)}{\|u\|} \ge \|u\|^{p^- -1} \to \infty, \quad \text{as } \|u\| \to \infty.$

$$T_1^{-1}(f_n) = u_n \text{ and } T_1^{-1}(f) = u.$$

By the coercivity of T_1, we deduce that the sequence $(u_n)_n$ is bounded in the reflexive space $W_0^{2,p(.)}(\Omega)$. For a subsequence if necessary, we have $u_n \rightharpoonup \widehat{u}$ in $W_0^{2,p(.)}(\Omega)$, for a some $\widehat{u}$. Then

$$\lim_{n\to+\infty}\langle T_1(u_n) - T_1(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty}\langle f_n - f, u_n - \widehat{u}\rangle = 0.$$

It follows by the second assertion and the continuity of T_1 that

$$u_n \to \widehat{u} \text{ in } W_0^{2,p(x)}(\Omega) \text{ strongly and}$$
$$T_1(u_n) \to T_1(\widehat{u}) = T_1(u) \text{ in } W^{-2,p'(x)}(\Omega)$$

Further , since T_1 is an into operator, we conclude that $u \equiv \widehat{u}$.

This completes the proof.

Lemma 2.7 *[13] The following statements hold*

(i) $-\triangle_{p(x)} := T_2$ *is continuous, bounded and strictly monotone.*

(ii) $-\triangle_{p(x)} := T_2$ *is of* (S_+) *type.*

(iii) $-\triangle_{p(x)} := T_2$ *is a homeomorphism.*

The following lemma plays a central key to prove our main result related to the existence.

Lemma 2.8 *We have the following statements*

(i) φ' *is completely continuous.*

(ii) *The functional* Φ *satisfies the Palais-Smale condition on* $\mathcal{M}$, *i.e., for* $\{u_n\} \subset \mathcal{M}$, *if* $\{\Phi(u_n)\}_n$ *is bounded and*

$$\Phi'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

$\{u_n\}$ *has a convergent subsequence in* X.

Proof (i) First let us prove that φ' is well defined. Let $u, v \in X$. We have

$$\langle \varphi'(u), v \rangle = \int_\Omega |u|^{p(x)-1} v \, dx.$$

By applying Hölder's inequality, we obtain

$$\langle \varphi'(u), v \rangle \leq |u|_{p(x)}^{p(x)-1} |v|_{p(x)}$$

Then

$$|\langle \varphi'(u), v \rangle| \leq C \|u\|^{p(x)-1} \|v\|,$$

where C is the constant given by the embedding of $W_0^{2,p(.)}(\Omega)$ in $L^{p(.)}(\Omega)$. Hence

$$\|\varphi'(u)\|_* \leq C \|u\|^{p(x)-1},$$

where $\|.\|_*$ is the dual norm associated with $\|.\|$.
For the complete continuity of φ', we argue as follow. Let $(u_n)_n \subset X$ be a bounded sequence and $u_n \rightharpoonup u$ (weakly) in X. Due the fact that the embedding $W_0^{2,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is compact u_n converges strongly to u in $L^{p(.)}(\Omega)$, and there exists a positive function $g \in L^{p(.)}(\Omega)$ such that

$$| u | \leq g \text{ a.e. in } \Omega.$$

Since $g \in L^{p(.)-1}(\Omega)$, it follows from the Dominated Convergence Theorem that

$$| u_n |^{p(x)-2} u_n \to | u |^{p(x)-2} u \text{ in } L^{p'(.)}(\Omega)$$

That is,

$$\varphi'(u_n) \to \varphi'(u) \text{ in } L^{p'(.)}(\Omega).$$

Recall that the embedding

$$L^{p'(.)}(\Omega) \hookrightarrow W^{-2,p'(.)}(\Omega)$$

is compact. Thus

$$\varphi'(u_n) \to \varphi'(u) \text{ in } X^*$$

This proves the assertion (i).
(ii) by the definition of Φ we have

$$\Phi(u) \geq \frac{1}{p^+} \|u\|^{p^-},$$

then u_n is bounded in X. So we deduce that there exists a subsequence, again denoted $\{u_n\}$, and $u \in X$ such that $\{u_n\}$ converges weakly to u in X. On the other hend, by (2.8) we get

$$\lim_{n\to\infty} \langle \Phi'(u_n), (u_n - u) \rangle = 0. \tag{2.9}$$

Then, by (ii) of lemma 2.6 and (ii) of lemma 2.7, we conclude that $\{u_n\}$ converges strongly to $u \in X$. This achieves the proof the lemma.

3 Existence results

Set

$$\Gamma_j = \{K \subset \mathcal{M} : K \text{ symmetric, compact and } \gamma(K) \geq j\},$$

where $\gamma(K) = j$ being the Krasnoselskii's genus of set K, i.e., the smallest integer j, such that there exists an odd continuous map from K to $\mathbb{R}^j \setminus \{0\}$.

Now, let us establish some useful properties of Krasnoselskii genus proved by Szulkin [12].

Lemma 3.1 *Let* X *be a real Banach space and* A, B *be symmetric subsets of* $E \setminus \{0\}$ *which are closed in* X. *Then*

(a) *If there exists an odd continuous mapping* $f : A \to B$, *then* $\gamma(A) \leq \gamma(B)$

(b) *If* $A \subset B$ *then* $\gamma(A) \leq \gamma(B)$.

(c) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(d) *If* $\gamma(B) < +\infty$ *then* $\gamma(\overline{A - B} \geq \gamma(A) - \gamma(B)$.

(e) *If* A *is compact then* $\gamma(A) < +\infty$ *and there exists a neighborhood* $\mathcal{N}$ *of* A, N *is a symmetric subset of* $X \setminus \{0\}$, *closed in* X *such that* $\gamma(N) = \gamma(A)$.

(f) *If* $\mathcal{N}$ *is a symmetric and bounded neighborhood of the origin in* $\mathbb{R}^k$ *and if* A *is homeomorphic to the boundary of* $\mathcal{N}$ *by an odd homeomorphism then* $\gamma(A) = k$.

(g) *If* X_0 *is a subspace of* X *of codimension* k *and if* $\gamma(A) > k$ *then* $A \cap X_0 \neq \phi$.

Let us now state the first our main result of this paper using Ljusternick-Schnirelmann theory:

Theorem 3.2 *For any integer* $j \in \mathbb{N}^*$,

$$\lambda_j = \inf_{K \in \Gamma_j} \max_{u \in K} \Phi(u),$$

is a critical value of Φ *restricted on* $\mathcal{M}$. *More precisely, there exist* $u_j \in K$, *such that*

$$\lambda_j = \Phi(u_j) = \sup_{u \in K} \Phi(u),$$

and u_j *is a solution of* (1.1) *associated to positive eigenvalue* λ_j. *Moreover,*

$$\lambda_j \to \infty, \text{ as } j \to \infty$$

Proof We only need to prove that for any $j \in \mathbb{N}^*$, $\Gamma_j \neq \emptyset$ and the last assertion. Indeed, since $W_0^{2,p(.)}(\Omega)$ is separable, there exists $(e_i)_{i\geq 1}$ linearly dense in $W_0^{2,p(.)}(\Omega)$ such that
$\operatorname{supp} e_i \cap \operatorname{supp} e_n = \emptyset$ if $i \neq n$. We may assume that $e_i \in \mathcal{M}$ (if not, we take $e_i' \equiv \frac{e_i}{[p(x)\varphi(e_i)]^{\frac{1}{p(x)}}}$).
Let now $j \in \mathbb{N}^*$ and denote

$$F_j = \operatorname{span}\{e_1, e_2, \ldots, e_j\}$$

Clearly, F_j is a vector subspace with $\dim F_j = j$. If $v \in F_j$, then there exist $\alpha_1, \ldots \alpha_j$ in $\mathbb{R}$, such that
$v = \sum_{i=1}^{j} \alpha_i e_i.$
Thus

$$\varphi(v) = \sum_{i=1}^{j} |\alpha_i|^{p(.)} \varphi(e_i) = \sum_{i=1}^{j} |\alpha_i|^{p(.)}.$$

It follows that the map

$$v \mapsto (\varphi(v))^{\frac{1}{p(.)}} = |||v|||$$

defines a norm on F_j. Consequently, there is a constant $c > 0$ such that

$$c\|v\| \leq |||v||| \leq \frac{1}{c}\|v\|.$$

This implies that the set

$$\mathcal{V}_j = F_j \cap \left\{ v \in W_0^{2,p(.)}(\Omega) : \varphi(v) \leq 1 \right\},$$

is bounded because $\mathcal{V}_k \subset B(0, \frac{1}{c})$, where

$$B\left(0, \frac{1}{c}\right) = \left\{ u \in W_0^{2,p(.)}(\Omega), \text{ such that} \|u\| \leq \frac{1}{c} \right\}.$$

Thus, $\mathcal{V}_j$ is a symmetric bounded neighborhood of $0 \in F_j$. Moreover, $F_j \cap \mathcal{M}$ is a compact set. By (f) of Lemma 2.7, we conclude that $\gamma(F_j \cap \mathcal{M}) = j$ and then we obtain finally that $\Gamma_j \neq \emptyset$. This completes the proof of first part of the theorem.
Now, we claim that

$$\lambda_j \to \infty, \text{ as } j \to \infty$$

Let $(e_k, e_n^*)_{k,n}$ be a bi-orthogonal system such that $e_k \in W_0^{2,p(.)}(\Omega)$ and $e_n^* \in W^{-2,p'(.)}(\Omega)$, the $(e_k)_k$ are linearly dense in $W_0^{2,p(.)}(\Omega)$ and the $(e_n^*)_n$ are total for the dual $W^{-2,p'(.)}(\Omega)$). For $k \in \mathbb{N}^*$, set

$$F_k = \operatorname{span}\left\{e_1, \ldots, e_k\right\} \text{ and } F_k^{\perp} = \operatorname{span}\left\{e_{k+1}, e_{k+2}, \ldots\right\}.$$

By (g) of Lemma 2.7, we have for any $K \in \Gamma_k$, $K \cap F_{k-1}^{\perp} \neq \emptyset$. Thus

$$t_k = \inf_{K \in \Gamma_k} \sup_{u \in K \cap F_{k-1}^{\perp}} \Phi(u) \to \infty, \text{ as } k \to \infty$$

Indeed, if not, for k is large, there exists $u_k \in F_{k-1}^{\perp}$ with $|u_k|_{p(.)} = 1$ such that

$$t_k \leq \Phi(u_k) \leq M,$$

for some $M > 0$ independent of k. Thus $\|u_k\|_{p(.)} \leq M$. This implies that $(u_k)_k$ is bounded in X. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converges weakly in X and strongly in $L^{p(.)}(\Omega)$. By our choice of $F_{k-1}^{\perp}$, we have $u_k \rightharpoonup 0$ weakly in X, because $\langle e_n^*, e_k \rangle = 0$, for any $k > n$. This contradicts the fact that $|u_k|_{p(.)} = 1$ for all k. Since $\lambda_k \geq t_k$ the claim is proved.

Corrolary 3.3 *we have the following statements:*

(i) $\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\,dx, u \in X, \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx = 1\right\}.$,

(ii) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$,

(iii) $\lambda_1 = Inf\,\Lambda$ *(i.e.,* λ_1 *is the smallest eigenvalue in the spectrum of* (1.1)*).*

Proof

(i) For $u \in \mathcal{M}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, Φ is even and that

$$\Phi(u) = \max_{K_1} \Phi \geq \inf_{K \in \Gamma_1} \max_{u \in K} \Phi(u).$$

Thus

$$\inf_{u \in \mathcal{M}} \Phi(u) \geq \inf_{K \in \Gamma_1} \max_{u \in K} \Phi(u) = \lambda_1.$$

On the other hand, $\forall K \in \Gamma_1$, $\forall u \in K$, we have

$$\sup_{u \in K} \Phi \geq \Phi(u) \geq \inf_{u \in \mathcal{M}} \Phi(u).$$

It follows that

$$\inf_{K \in \Gamma_1} \max_{K} \Phi = \lambda_1 \geq \inf_{u \in \mathcal{M}} \Phi(u).$$

Then

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\,dx, u \in X, \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx = 1\right\}.$$

(ii) For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\lambda_i \geq \lambda_j$. As regards to $\lambda_n \to \infty$, it is proved before in Theorem 3.2.

(iii) Let $\lambda \in \Lambda$. Thus there exists u_λ an eigenfunction of λ such that

$$\int_\Omega \frac{1}{p(x)}|u_\lambda|^{p(x)}\,dx = 1.$$

Therefore

$$\Delta^2_{p(x)} u_\lambda - \Delta_{p(x)} u_\lambda = \lambda |u_\lambda|^{p(x)-2} u_\lambda \text{ in } \Omega.$$

Then

$$\int_\Omega \frac{1}{p(x)}(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)})\,dx = \lambda \int_\Omega \frac{1}{p(x)}|u_\lambda|^{p(x)}\,dx.$$

In view of the characterization of λ_1 in (2.1), we conclude that

$$\lambda = \frac{\int_\Omega \frac{1}{p(x)}(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)})\,dx}{\int_\Omega \frac{1}{p(x)}|u_\lambda|^{p(x)}\,dx} = \int_\Omega \frac{1}{p(x)}(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)})\,dx \geq \lambda_1.$$

This implies that $\lambda_1 = \inf \Lambda$.

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