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Degenerate $p(x)$-elliptic equation with second membre in L^1

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ABSTRACT

In this paper, we prove the existence of a solution of the strongly nonlinear degenerate $p(x)$-elliptic equation of type:

$$(\mathcal{P})\begin{cases} -div\, a(x,u,\nabla u)+g(x,u,\nabla u) &= f \quad in\,\Omega, \\ u &= 0 \quad on \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $I\!R^N$, $N\geq 2$, a is a Carathéodory function from $\Omega\times I\!R\times I\!R^N$ into $I\!R^N$, who satisfies assumptions of growth, ellipticity and strict monotonicity. The nonlinear term g: $\Omega\times I\!R\times I\!R^N\longrightarrow I\!R$ checks assumptions of growth, sign condition and coercivity condition, while the right hand side f belongs to $L^1(\Omega)$.

1 Introduction

Let Ω be a bounded open subset of $I\!R^N$, $N\geq 2$, let $\partial\Omega$ its boundary and $p(x)\in C(\overline{\Omega})$ with $p(x)>1$.
Let ν be a weight function in Ω, ie: ν measurable and strictly positive a.e. in Ω. We suppose furthermore, that the weight function satisfies also the integrability conditions defined in section 2.
Let us consider the following degenerate $p(x)$-elliptic problem with boundary condition

$$(\mathcal{P})\begin{cases} Au+g(x,u,\nabla u) &= f \quad \text{in}\,\Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where A is a Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega,\nu)$ to its dual $W^{-1,p'(x)}(\Omega,\nu^*)$, with $\nu^*=\nu^{1-p'(x)}$, by :

$$Au=-\text{div}\, a(x,u,\nabla u).$$

where a is a Carathéodory function from $\Omega\times I\!R\times I\!R^N\longrightarrow I\!R^N$ who satisfies assumptions of growth, ellipticity and strict monotonicity, while the nonlinear term g: $\Omega\times I\!R\times I\!R^N\longrightarrow I\!R$ checks assumptions of sign and growth. We suppose moreover that g checks the following condition of coercivity:

$$\begin{cases} \exists\,\rho_1>0,\exists\,\rho_2>0 \quad \text{such that :} \\ \text{for } |s|\geq\rho_1,\ |g(x,s,\xi)|\geq\rho_2\,\nu(x)|\xi|^{p(x)} \end{cases}$$

We suppose also that the second member f belongs to $L^1(\Omega)$.

Let consider the following degenerate $p(x)$-elliptic problem of Dirichlet:

$$\begin{cases} Au+g(x,u,\nabla u) &= f \quad \text{in } \mathcal{D}'(\Omega), \\ u\in W_0^{1,p(x)}(\Omega,\nu), & g(x,u,\nabla u)\in L^1(\Omega). \end{cases} \tag{1}$$

In the case where p is constant and without weight, there is a wide literature in which one can find existence results for problem (1). When the second member f belongs to $W^{-1,p'}(\Omega)$, A. Bensoussan, L. Boccardo and F. Murat [6] studied the problem and give an existence result. While if $f\in L^1(\Omega)$ the initiated basic works were given by H. Brezis and Strauss [9], L. Boccardo and T. Gallouët [7] also proved an existence result for (1), which was extended to the a unilateral case studied by A. Benkirane and A. Elmahi [5]. When g is not necessarily the null function, T. Del Vecchio [10] proved first existence result for problem (1) in the case where g does not depend on the gradient and then in V. M Monetti and L. Randazzo [16] using, in both works, the rearrangement techniques.
Whoever in [1], Y. Akdim, E. Azroul and A. Benkirane treated the problem (1) within the framework of Sobolev spaces with weight $W_0^{1,p}(\Omega,\omega)$, but while keeping p constant.
E. Azroul, A. Barbara and H. Hjiej [2] studied (1), in the nonclassical case by considering nonstandard Sobolev spaces without weight $W_0^{1,p(x)}(\Omega)$. See as well [3] where existence and regularity of entropy solutions was obtained for equation (1) with degenerated

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second member.
Our objectif, in this paper, is to study equation (1) by adopting Sobolev spaces with weight $\nu(x)$, and to variable exponents $p(x)$, $W_0^{1,p(x)}(\Omega,\nu)$. We prove that the problem (1) admits at least a solution $u \in W_0^{1,p(x)}(\Omega,\nu)$.

2 Functional frame

Throughout this section, we suppose that the variable exponent $p(\cdot):\overline{\Omega}\to[1,+\infty[$ is log-Hölder continuous on Ω, that is there is a real constant $c>0$ such that $\forall x,y\in\overline{\Omega}$, $x\neq y$ with $|x-y|<1/2$ one has:

$$|p(x)-p(y)| \leq \frac{c}{-log|x-y|}$$

and satisfying

$$p^- \leq p(x) \leq p^+ < +\infty$$

where

$$p^- := \operatorname{ess}\inf_{x\in\overline{\Omega}} p(x); \quad p^+ := \operatorname{ess}\sup_{x\in\overline{\Omega}} p(x).$$

We define

$$C_+(\overline{\Omega}) = \{h \,\text{log-Hölder continous on } \overline{\Omega}, h(x)>1\}.$$

Definition 1 *Let ν be a function defined in Ω; we call ν a weight function in Ω if it is measurable and strictly positive a.e. in Ω.*

2.1 Lebesgue spaces with weight and to variable exponents

Let $p\in C_+(\overline{\Omega})$ and ν be a weighted function in Ω.
We define the Lebesgue space with weight and to variable exponents $L^{p(x)}(\Omega,\nu)$, by
$L^{p(x)}(\Omega,\nu) = \{u:\Omega\to \mathbb{R}$, measurable : $\int_\Omega \nu(x)|u|^{p(x)}dx<\infty\}$, equipped with the Luxemburg norm:

$$\|u\|_{p(x),\nu} = \inf\left\{\mu>0 : \int_\Omega \nu(x)|\frac{u}{\mu}|^{p(x)}dx \leq 1\right\}$$

Proposition 1 *The space $\left(L^{p(x)}(\Omega,\nu), \|.\|_{p(x),\nu}\right)$ is of Banach.*

Proof:
Considering the operator

$$M_{\nu^{\frac{1}{p(x)}}} : L^{p(x)}(\Omega,\nu) \longrightarrow L^{p(x)}(\Omega), f \to M_{\nu^{\frac{1}{p(x)}}}(f) = f\,\nu^{\frac{1}{p(x)}}$$

It's clear that $M_{\nu^{\frac{1}{p(x)}}}$ is isomorphism from $L^{p(x)}(\Omega,\nu)$ into $L^{p(x)}(\Omega)$, then $M_{\nu^{\frac{1}{p(x)}}}$ is a continuous biuniformly application. Seeing that $L^{p(x)}(\Omega)$ is a Banach space, then $\left(L^{p(x)}(\Omega,\nu), \|.\|_{p(x),\nu}\right)$ is of Banach.

Let's note $\rho_\nu(u) = \int_\Omega \nu(x)|u|^{p(x)}dx$.

Remark 1 *In simple case $\nu(x)=1$, we find again the Lebesgue space with variable exponents $L^{p(x)}(\Omega)$; and $\rho_\nu(u)=\rho_1(u):=\rho(u)=\int_\Omega |u|^{p(x)}dx$, (see [12],[13] and [17])*

Lemma 1 *For all function $u\in L^{p(x)}(\Omega,\nu)$. There are the following assertions:*
(i) $\rho_\nu(u)>1 \quad (=1;<1) \Leftrightarrow \|u\|_{p(x),\nu}>1 \quad (=1;<1)$, *respectively.*
(ii) *If* $\|u\|_{p(x),\nu}>1$ *then* $\|u\|_{p(x),\nu}^{p_-} \leq \rho_\nu(u) \leq \|u\|_{p(x),\nu}^{p^+}$.
(iii) *If* $\|u\|_{p(x),\nu}<1$ *then* $\|u\|_{p(x),\nu}^{p^+} \leq \rho_\nu(u) \leq \|u\|_{p(x),\nu}^{p_-}$.

Proof:
Seeing that $\rho_\nu(u)=\rho(\nu^{\frac{1}{p(x)}}u)$ and $\|\nu^{\frac{1}{p(x)}}u\|_{p(x)} = \|u\|_{p(x),\nu}$, and using [17], we prove the lemma 2.1 above.
Let ν be a weight function such that the following condition:

$$\textbf{(w1)}\ \nu\in L^1_{loc}(\Omega);\ \nu^{\frac{-1}{p(x)-1}}\in L^1_{loc}(\Omega).$$

Proposition 2 *Let Ω be a bounded open subset of $\mathbb{R}^N$, and ν be a weight function on Ω,*
*If **(w1)** is verified then $L^{p(x)}(\Omega,\nu)\hookrightarrow L^1_{loc}(\Omega)$.*

Proof:
Let K be a included compact on Ω. Using Hölder inequality we have

$$\begin{aligned}\int_K |u|dx &= \int_K |u|\nu^{\frac{1}{p(x)}}\nu^{\frac{-1}{p(x)}}dx \\ &\leq 2\||u|\nu^{\frac{1}{p(x)}}\|_{L^{p(x)}(K)}\|\nu^{\frac{-1}{p(x)}}\|_{L^{p'(x)}(K)}, \\ &\leq 2\|u\|_{p(x),\nu}\left(\int_K \nu^{\frac{-p'(x)}{p(x)}}dx+1\right)^{\frac{1}{p'_-}}, \\ &\leq 2\|u\|_{p(x),\nu}\left(\int_K \nu^{\frac{-1}{p(x)-1}}dx+1\right)^{\frac{1}{p'_-}}.\end{aligned}$$

Thanks to the assumption **(w1)** we deduce that $\int_K |u|dx \leq C\|u\|_{p(x),\nu}$.

2.2 Spaces of Sobolev with weight and to variable exponents

Let $p\in C_+(\overline{\Omega})$ and ν be a weight function in Ω.
We define the space of Sobolev with weight and to variable exponents denoted $W^{1,p(x)}(\Omega,\overrightarrow{\nu})$, by

$$W^{1,p(x)}(\Omega,\nu) = \left\{u\in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i}\in L^{p(x)}(\Omega,\nu), i=1,...,N\right\},$$

equipped with the norm

$$\|u\|_{1,p(x),\nu} = \|u\|_{p(x)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{p(x),\nu}$$

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which is equivalent to the Luxemburg norm

$$|||u||| = \inf\left\{\mu > 0 : \int_\Omega \left(|\frac{u}{\mu}|^{p(x)} + \nu(x)\sum_{i=1}^{N} |\frac{\frac{\partial u}{\partial x_i}}{\mu}|^{p(x)}\right)dx \le 1\right\}.$$

Proposition 3 *Let ν be a weight function in Ω who checks the condition* ***(w1)****.*
Then the space $\left(W^{1,p(x)}(\Omega,\nu), \|.\|_{1,p(x),\nu}\right)$ is of Banach.

Proof:
Let us consider $(u_n)_n$ a Cauchy sequence of $\left(W^{1,p(x)}(\Omega,\nu), \|.\|_{1,p(x),\nu}\right)$.
Then $(u_n)_n$ is a Cauchy sequence of $L^{p(x)}(\Omega)$ and the sequence $(\frac{\partial u_n}{\partial x_i})_n$ is also a Cauchy belong $L^{p(x)}(\Omega,\nu)$, $i = 1,...,N$.
According the proposition 2.1, there exists $u \in L^{p(x)}(\Omega)$ such that $u_n \to u$ in $L^{p(x)}(\Omega)$, and there exists $v_i \in L^{p(x)}(\Omega,\nu)$ such that $\frac{\partial u_n}{\partial x_i} \to v_i$ in $L^{p(x)}(\Omega,\nu)$, $i = 1,...,N$.
Seeing that proposition 2.2, we have $L^{p(x)}(\Omega,\nu) \subset L^1_{loc}(\Omega)$ and $L^1_{loc}(\Omega) \subset D'(\Omega)$.
Thus, we obtain $\forall \varphi \in D(\Omega)$,

$$\begin{aligned}\langle T_{v_i}, \varphi\rangle &= \lim_{n\to\infty}\langle T_{\frac{\partial u_n}{\partial x_i}}, \varphi\rangle,\\ &= -\lim_{n\to\infty}\langle T_{u_n}, \frac{\partial \varphi}{\partial x_i}\rangle,\\ &= -\langle T_u, \frac{\partial \varphi}{\partial x_i}\rangle,\\ &= \langle T_{\frac{\partial u}{\partial x_i}}, \varphi\rangle.\end{aligned}$$

Hence $T_{v_i} = T_{\frac{\partial u}{\partial x_i}}$, i.e. $v_i = \frac{\partial u}{\partial x_i}$.
Consequently

$$u \in W^{1,p(x)}(\Omega,\nu),$$

and

$$u_n \to u \text{ in } W^{1,p(x)}(\Omega,\nu),$$

We deduce then that $W^{1,p(x)}(\Omega,\nu)$ is a complete space.
• On another side, seeing that ν satisfies the condition **(w1)**, we prove that $C_0^\infty(\Omega)$ is included in $W^{1,p(x)}(\Omega,\nu)$; that enables us to define the following space

$$W_0^{1,p(x)}(\Omega,\nu) = \overline{C_0^\infty(\Omega)}^{\|.\|_{1,p(x),\nu}},$$

who is closed in complete space, then it's complete.

Proposition 4 *(Characterization of the dual space $(W_0^{1,p(x)}(\Omega,\nu))^*$)*
Let $p(.) \in C_+(\overline{\Omega})$ and ν a vector of weight who satisfies the condition ***(w1)****. Then for all $G \in (W_0^{1,p(x)}(\Omega,\nu))^*$, there exist a unique system of functions $(g_0, g_1, ..., g_N) \in L^{p'(x)}(\Omega)\times(L^{p'(x)}(\Omega,\nu^{1-p'(x)}))^N$ such that $\forall f \in W_0^{1,p(x)}(\Omega,\nu)$:*

$$G(f) = \int_\Omega f(x)g_0(x)dx + \sum_{i=1}^{N}\int_\Omega \frac{\partial f}{\partial x_i} g_i(x)dx.$$

Proof
The proof of this proposition is similar to that of [14] (theorem3.16).
Besides the **(w1)** assumption, we suppose that the function weight satisfied

$$\textbf{(w2)}\ \ \nu^{-s(x)} \in L^1_{loc}(\Omega)$$

where s is a positive function to specify afterwards.
Let us introduce the function p_s defined by

$$p_s(x) = \frac{p(x)s(x)}{s(x)+1},$$

we have

$$p_s(x) < p(x) \text{ a.e. in } \Omega.$$

and

$$\begin{cases} p_s^*(x) &= \frac{Np_s(x)}{N-p_s(x)} = \frac{Np(x)s(x)}{N(s(x)+1)-p(x)s(x)} \text{ if } p(x)s(x) < N(s(x)+1),\\ p_s^*(x) & \text{arbitrary, else if,}\end{cases}$$

Proposition 5 *Let $p, s \in C_+(\overline{\Omega})$ and ν a function weight which satisfies* ***(w1)*** *and* ***(w2)****. Then $W^{1,p(x)}(\Omega,\nu) \hookrightarrow W^{1,p_s(x)}(\Omega)$.*

Proof
According to the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |v(x)|^{p_s(x)}dx &= \int_\Omega |v(x)|^{p_s(x)} \nu^{\frac{p_s(x)}{p(x)}} \nu^{\frac{-p_s(x)}{p(x)}} dx\\ \le& \left(\frac{1}{(\frac{p}{p_s})_-} + \frac{1}{(s+1)^-}\right)\||v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\|_{\frac{p(x)}{p_s(x)}} \|\nu^{\frac{-p_s(x)}{p(x)}}\|_{s(x)+1}\\ \le& C_0\left(\int_\Omega |v(x)|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}\left(\int_\Omega \nu(x)^{-s(x)}dx\right)^{\frac{1}{\gamma_2}}\\ \le& C_0\left(\int_\Omega |v(x)|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}\left(\int_\Omega \nu(x)^{-s(x)}dx\right)^{\frac{1}{\gamma_2}}\\ \le& C_0C_1\left(\int_\Omega |v(x)|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}, \text{ according to } \textbf{(w2)}.\end{aligned}$$

If we take $v = \frac{\partial u}{\partial x_i}$, we will obtain then

$$\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_s(x)}dx \le C_0C_1\left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}$$

where

$$\gamma_1 = \begin{cases} (\frac{p}{p_s})_- & \text{if} \quad \||v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\|_{\frac{p(x)}{p_s(x)}} \ge 1,\\ (\frac{p}{p_s})^+ & \text{if} \quad \||v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\|_{\frac{p(x)}{p_s(x)}} < 1,\end{cases}$$

consequently

$$\begin{aligned}\|\frac{\partial u}{\partial x_i}(x)\|^{\gamma_2}_{p_s(x)} \le& \ C_0C_1\left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}\\ \le& \ C_0C_1\|\frac{\partial u}{\partial x_i}(x)\|^{\frac{\gamma_3}{\gamma_1}}_{p(x),\nu}\end{aligned}$$

where

$$\gamma_2 = \begin{cases} (p_s)_- & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p_s(x)} \ge 1,\\ (p_s)^+ & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p_s(x)} < 1,\end{cases}$$

and

$$\gamma_3 = \begin{cases} p^+ & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu} \geq 1, \\ p_- & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu} < 1. \end{cases}$$

thus

$$\|\frac{\partial u}{\partial x_i}\|_{p_s(x)} \leq C_0 C_1 \|\frac{\partial u}{\partial x_i}\|_{p(x),\nu}^{\frac{\gamma_3}{\gamma_1\gamma_2}}, \ i = 1,2,...,N. \quad (2)$$

Seeing that

$$p_s(x) < p(x) \ \text{a.e. in} \ \Omega.$$

then, there is a constant

$$C > 0 \ \text{such that} \ \|u\|_{L^{p_s(x)}(\Omega)} \leq C\|u\|_{L^{p(x)}(\Omega)},$$

we conclude then that

$$W^{1,p(x)}(\Omega,\nu) \hookrightarrow W^{1,p_s(x)}(\Omega).$$

Corollaire 1 *Let $p, s \in C_+(\overline{\Omega})$ and ν a fuction weight which satisfies **(w1)** and **(w2)**. Then $W^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$, for $1 \leq r(x) < p_s^*(x)$*

3 Basic assumption

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$, let $p(.) \in C_+(\overline{\Omega})$, and ν a function weight in Ω such that:

$$\nu \in L^1_{loc}(\Omega) \quad (3)$$

$$\nu^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega) \quad (4)$$

and

$$\nu^{-s(x)} \in L^1_{loc}(\Omega) \ \text{where} \ s(x) \in \left]\frac{N}{p(x)}, \infty\right[\cap \left[\frac{1}{p(x)-1}, \infty\right[\quad (5)$$

Let A Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega,\nu)$ to its dual $W^{-1,p'(x)}(\Omega,\nu^*)$ by

$$Au = -\text{div}\, a(x,u,\nabla u),$$

where a is a Carathéodory function satisfying the following assumptions:

$$|a_i(x,r,\zeta)| \leq \beta\nu^{\frac{1}{p(x)}}\left[b(x) + |r|^{\frac{p(x)}{p'(x)}} + \nu^{\frac{1}{p'(x)}}|\zeta|^{p(x)-1}\right], i = 1,..,N. \quad (6)$$

$$[a(x,r,\zeta) - a(x,r,\overline{\zeta})](\zeta - \overline{\zeta}) > 0, \quad \forall\, \zeta \neq \overline{\zeta} \in I\!R^N. \quad (7)$$

$$a(x,s,\zeta)\zeta \geq \alpha\nu|\zeta|^{p(x)}. \quad (8)$$

with $b(x)$ be a positive function in $L^{p'(x)}(\Omega)$, and α, β are two strictly positive constants. On another side, let g: $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ a Carathéodory function who satisfied a.e. for $x \in \Omega$ and for all $r \in I\!R$, $\zeta \in I\!R^N$ the following conditions:

$$g(x,r,\zeta)r \geq 0. \quad (9)$$

$$|g(x,r,\zeta)| \leq d(|r|)(\nu(x)|\zeta|^{p(x)} + c(x)). \quad (10)$$

with $d : I\!R^+ \longrightarrow I\!R^+$ a continuous, nondecreasing and positive function; whereas $c(x)$ be a positive function in $L^1(\Omega)$.
Moreover, we suppose that g checks:

$$\begin{cases} \exists\, \rho_1 > 0, \exists\, \rho_2 > 0 \,\text{such that}: \\ \text{for}\ |s| \geq \rho_1,\ |g(x,s,\xi)| \geq \rho_2\nu(x)|\xi|^{p(x)} \end{cases} \quad (11)$$

and

$$f \in L^1(\Omega). \quad (12)$$

We have the following theorem

Theorem 1 *Assume that $(3)-(12)$ holds, then the problem (1) admits at least one solution $u \in W_0^{1,p(x)}(\Omega,\nu)$*

Remark 2 *If $f \in W^{-1,p'(x)}(\Omega,\nu^*)$ then we have $ug(x,u,\nabla u) \in L^1(\Omega)$.*
But if $f \in L^1(\Omega)$, we necessarily do not have $ug(x,u,\nabla u) \in L^1(\Omega)$.

Counter example:
Let us consider $\Omega = \{x \in I\!R^2 : |x| < 1\}$, the ball open unit of $I\!R^2$, let $\lambda \in [\frac{1}{4}, \frac{1}{3}[$, we then have $u(x) = ln^{\lambda}(\frac{1}{|x|}) \in H_0^1(\Omega)$ and $-\Delta u \in L^1(\Omega)$, such that $-\Delta u \geq 0, u|\nabla u|^2 \in L^1(\Omega)$ and $|u|^2|\nabla u|^2$ does not belong to $L^1(\Omega)$ see [7].

Remark 3 *The result of theorem 1 is not true when $g(x,r,\zeta) = 0$, seeing that in the non-degenerate case with $p(x) = p$ constant, $p \leq N$, and when $f \in L^1(\Omega)$, a solution of $Au = f$ does not belong to $W_0^{1,p}(\Omega)$ but belongs to $\bigcap_{1<q<\frac{N(p-1)}{N-1}} W_0^{1,q}(\Omega)$, see[8].*

Definition 2 *Let X a Banach space. An operator A from X to its dual X^* is called as type (M) if for all sequence $(u_n)_n \subset X$ satisfying:*

$$\begin{cases} (i) & u_n \rightharpoonup u \ \text{weakly in}\ X \\ (ii) & Au_n \rightharpoonup \chi \ \text{weakly in}\ X^* \\ (iii) & \limsup_{n\to\infty}\langle Au_n, u_n\rangle \leq \langle \chi, u\rangle, \end{cases}$$

then $\chi = Au$.

Remark 4 *The theorem 2.1 (p.171 [15]) remains valid if we replace A monotonous by: A of type (M).*

4 Approximate problem

Let $(f_n)_n$ a sequence of regularly functions such that $f_n \to f$ in $L^1(\Omega)$ and $\|f_n\| \leq \|f\| = C_1$.
Let us consider the following approximate problem:

$$(\mathcal{P}_n)\begin{cases} Au_n + g_n(x,u_n,\nabla u_n) = f_n & \text{in} \quad \Omega, \\ u_n \in W_0^{1,p(x)}(\Omega,\nu), \end{cases}$$

where $g_n(x,r,\zeta) = \frac{g(x,r,\zeta)}{1+\frac{1}{n}g(x,r,\zeta)}\chi_{\Omega_n}$, with χ_{Ω_n} is the characteristic function of Ω_n where Ω_n is a sequence of compact subsets which is increasing towards Ω.

We have $g_n(x,r,\zeta)r \geq 0; |g_n(x,r,\zeta)| \leq |g(x,r,\zeta)|$ and $|g_n(x,r,\zeta)| \leq n$.
Let us consider $G_n : W_0^{1,p(x)}(\Omega,\nu) \longrightarrow W^{-1,p'}(\Omega,\nu^*)$ defined by

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)\nabla v dx, u,v \in W_0^{1,p(x)}(\Omega,\nu).$$

Proposition 6 *The operator G_n is bounded.*

Proof:
Let u and v in $W_0^{1,p(x)}(\Omega,\nu)$, according to the Hölder inequality, we have:

$$\begin{aligned}
\langle G_n u, v\rangle &= \int_\Omega g_n(x,u,\nabla u)\nabla v dx,\\
&\leq \int_\Omega |g_n(x,u,\nabla u)|\nu(x)^{\frac{-1}{p(x)}}\nabla v \nu(x)^{\frac{1}{p(x)}}dx\\
&\leq (\tfrac{1}{p_-}+\tfrac{1}{p'_-})\| |g_n(x,u,\nabla u)|\nu(x)^{\frac{-1}{p(x)}}\|_{p'(x)}\|\nabla v \nu(x)^{\frac{1}{p(x)}}\|_{p(x)},\\
&\leq C\left(\int_\Omega |g_n(x,u,\nabla u)|^{p'(x)}\nu(x)^* dx\right)^{\frac{1}{\gamma_1}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\\
&\leq C\left(\int_\Omega n^{p'(x)}\nu(x)^* dx\right)^{\frac{1}{\gamma_1}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\\
&\leq C n^{p'_+}\left(\int_\Omega \nu(x)^* dx\right)^{\frac{1}{\gamma_1}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\\
&\leq C'\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}
\end{aligned}$$

■

Proposition 7 *The operator $A+G_n : W_0^{1,p(x)}(\Omega,\nu) \longrightarrow W^{-1,p'(x)}(\Omega,\nu^*)$ defined by $\forall u,v \in W_0^{1,p(x)}(\Omega,\nu)$,*

$$\langle (A+G_n)u, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v dx + \int_\Omega g_n(x,u,\nabla u)\nabla v dx,$$

is bounded, coercif, hemicontinous and of type (M).

Thanks to [15], the approximate problem admits at least a solution.

Proof of the proposition 7
Using (6) and Hölder inequality, we conclude that A is bounded, by taking account of the proposition 6, we will obtain that $A+G_n$ is bounded. The coercivity rise from (8) and from (9). It remains to be shown that $A+G_n$ is hemicontinous, i.e. that:

$$\forall u,v,w \in W_0^{1,p(x)}(\Omega,\nu),$$

$$\langle (A+G_n)(u+tv), w\rangle \longrightarrow \langle (A+G_n)(u+t_0v), w\rangle, \text{when} t \to t_0.$$

seeing that for a.e. $x \in \Omega$, we have:

$$a_i(x,u+tv,\nabla(u+tv)) \longrightarrow a_i(x,u+t_0v,\nabla(u+t_0v)),\ t \to t_0,$$

then (6) combined to the lemma 2, implies that $a_i(x,u+tv,\nabla(u+tv)) \rightharpoonup a_i(x,u+t_0v,\nabla(u+t_0v))$, weakly in $(L^{p'(x)}(\Omega,\nu^*))^N$, when $t \to t_0$. Finally, $\forall w \in W_0^{1,p(x)}(\Omega,\nu)$, $\forall u,v,w \in W_0^{1,p(x)}(\Omega,\nu)$

$$\langle A(u+tv), w\rangle \longrightarrow \langle A(u+t_0v), w\rangle, \text{ when } t \to t_0.$$

On another side $g_n(x,u+tv,\nabla(u+tv)) \to g_n(x,u+t_0v,\nabla(u+t_0v))$, when $t \to t_0$ a.e. $x \in \Omega$, moreover

$$\int_\Omega |g_n(x,u+tv,\nabla(u+tv))|^{p'(x)}dx \leq (\frac{1}{n})^{p'_+}|\Omega| < \infty,$$

while using, still the lemma 2, we obtain:

$$g_n(x,u+tv,\nabla(u+tv)) \rightharpoonup g_n(x,u+t_0v,\nabla(u+t_0v))$$

weakly in $L^{p'(x)}(\Omega)$, when $t \to t_0$.
Seeing that $w \in L^{p'(x)}(\Omega)$, we will have:

$$\langle G_n(u+tv), w\rangle \longrightarrow \langle G_n(u+t_0v), w\rangle, \text{ when } t \to t_0.$$

Now, we will show that $A+G_n$ satisfies the property of type (M), a.e. that, for all sequence $(u_j)_j \subset W_0^{1,p(x)}(\Omega,\nu)$ checking:

$$\begin{cases}
(i) & u_j \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega,\nu)\\
(ii) & (A+G_n)u_j \rightharpoonup \chi \text{ weakly in } W^{-1,p'(x)}(\Omega,\nu^*)\\
(iii) & \limsup_{j\to\infty}\langle (A+G_n)u_j, u_j-u\rangle \leq 0,
\end{cases}$$

then $\chi = (A+G_n)u$.
Indeed:

$$\begin{aligned}
\int_\Omega g_n(x,u_j,\nabla u_j)(u_j-u)dx &\leq \|g_n(x,u_j,\nabla u_j)\|_{p'(x)}\|u_j-u\|_{p(x)}\\
&\leq C\left(\int_{\Omega,} |g_n(x,u_j,\nabla u_j)|^{p'(x)}dx\right)^{\frac{1}{\nu}}\|u_j-u\|_{p(x),\nu} \quad \text{Thus}\\
&\leq C(\tfrac{1}{n})^{p'_+}|\Omega|\|u_j-u\|_{p(x),\nu} \longrightarrow 0 \text{ when } j \to \infty.
\end{aligned}$$

$$\lim_{j\to\infty}\langle G_n u_j, u_j-u\rangle = 0.$$

consequently, according to (iii), we obtain

$$\limsup_{j\to\infty}\langle Au_j, u_j-u\rangle \leq 0,$$

and seeing that A is pseudo-monotonous [11], we deduce that

$$Au_j \rightharpoonup Au \text{ weakly in } W^{-1,p'(x)}(\Omega,\nu^*)$$

and $\lim_{j\to\infty}\langle Au_j, u_j-u\rangle = 0$.
On another side

$$\begin{aligned}
0 &= \lim_{j\to\infty}\int_\Omega a(x,u_j,\nabla u_j)\nabla(u_j-u)dx\\
&= \lim_{j\to\infty}\Big(\int_\Omega \Big(a(x,u_j,\nabla u_j)-a(x,u_j,\nabla u)\Big)\nabla(u_j-u)dx\\
&\quad + \int_\Omega a(x,u_j,\nabla u)\nabla(u_j-u)dx\Big).
\end{aligned}$$

Moreover we have $a(x,u_j,\nabla u) \longrightarrow a(x,u,\nabla u)$ strongly in $(L^{p'(x)}(\Omega,\nu^*))^N$, then

$$\lim_{j\to\infty}\int_\Omega a(x,u_j,\nabla u)\nabla(u_j-u)dx = 0.$$

Thus

$$\lim_{j\to\infty}\int_\Omega \Big(a(x,u_j,\nabla u_j)-a(x,u_j,\nabla u)\Big)\nabla(u_j-u)dx = 0.$$

Using the lemma 3, see blow, we conclude that

$$\nabla u_j \longrightarrow \nabla u \text{ a.e. in } \Omega, \text{ when } j \to \infty.$$

Consequently

$$g_n(x, u_j, \nabla u_j) \longrightarrow g_n(x, u, \nabla u) \text{ a.e. in } \Omega, \text{ when } j \to \infty.$$

Seeing that $|g_n(x, u_j, \nabla u_j)| \leq n \in L^{p'(x)}(\Omega)$, then according to the dominated convergence theorem of Lebesgue, we obtain:

$$g_n(x, u_j, \nabla u_j) \longrightarrow g_n(x, u, \nabla u) \text{ in } L^{p'(x)}(\Omega).$$

and seeing that $v \in L^{p(x)}(\Omega)$, then we have: $\int_\Omega g_n(x, u_j, \nabla u_j) v dx \longrightarrow \int_\Omega g_n(x, u, \nabla u) v dx$ when $j \to \infty$, that means

$$G_n u_j \rightharpoonup G_n u \text{ in } W^{-1,p'(x)}(\Omega, v^*) \text{ when } j \to \infty.$$

Finally

$Au_j + G_n u_j \rightharpoonup Au + G_n u = \chi$ in $W^{-1,p'(x)}(\Omega, v^*)$ when $j \to \infty$.

5 Technical lemmas

Lemma 2 *Let γ a function weight in Ω, $r(.) \in C_+(\overline{\Omega})$, $g \in L^{r(x)}(\Omega, \gamma)$ and $(g_n)_n \subset L^{r(x)}(\Omega, \gamma)$ such that $\|g_n\|_{r(x),\gamma} \leq C$.*
If $g_n \to g$ a.e. in Ω then $g_n \rightharpoonup g$ weakly in $L^{r(x)}(\Omega, \gamma)$.

Proof:

Let $n_0 \geq 1$, let us pose

$$E(n_0) = \left\{ x \in \Omega : |g_n(x) - g(x)| \leq 1, \forall n \geq n_0 \right\}.$$

We have

$$mes(E(n_0)) \to mes(\Omega), \text{ when } n_0 \to \infty.$$

Let

$$F = \left\{ \varphi_{n_0} \in L^{r'(x)}(\Omega, \gamma^*) : \varphi_{n_0} = 0 \text{ a.e. in } \Omega \setminus E(n_0) \right\}.$$

Let us show that F is dense in $L^{r'(x)}(\Omega, \gamma^*)$:

let $f \in L^{r'(x)}(\Omega, \gamma^*)$, let us pose:

$$f_{n_0}(x) = \begin{cases} f(x) & if\ x \in E(n_0), \\ 0 & if\ x \in \Omega \setminus E(n_0). \end{cases}$$

We have

$$\rho_{r'(x),\gamma^*}(f_{n_0}(x) - f(x)) = \int_\Omega |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx$$

$$= \int_{E(n_0)} |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx$$

$$+ \int_{\Omega \setminus E(n_0)} |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx,$$

$$= \int_{\Omega \setminus E(n_0)} |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx,$$

$$= \int_\Omega |f(x)|^{r'(x)} \gamma^* \chi_{\Omega \setminus E(n_0)}(x) dx.$$

Let us pose $\psi_{n_0} = |f(x)|^{r'(x)} \gamma^* \chi_{\Omega \setminus E(n_0)}(x)$. we have

$$\begin{cases} \psi_{n_0} & \to 0 \quad \text{a.e.in } \Omega, \\ & and \\ |\psi_{n_0}| & \leq |f(x)|^{r'(x)} \gamma^*, \end{cases}$$

according to the dominate convergence theorem, we will have

$$\rho_{r'(x),\gamma^*}(f_{n_0}(x) - f(x)) \to 0, \text{ when } n_0 \to \infty,$$

what implies that F is dense in $L^{r'(x)}(\Omega, \gamma^*)$.
let us show, now, that

$$\lim_{n \to \infty} \int_\Omega \varphi(x)(g_n(x) - g(x)) dx = 0, \ \forall \varphi \in F.$$

Seeing that $\varphi = 0$ on $\Omega \setminus E_{n_0}$, it is thus enough to prove that

$$\lim_{n \to \infty} \int_{E_{n_0}} \varphi(x)(g_n(x) - g(x)) dx = 0.$$

Let us pose $\phi_n = \varphi(g_n - g)$, we have

$$\begin{cases} |\varphi(x)||(g_n(x) - g(x))| & \leq |\varphi(x)| \quad \text{in } E_{n_0}, \\ & and \\ \phi_n \longrightarrow 0, & \text{a.e. in } \Omega \end{cases}$$

According to the dominate convergence theorem, we have

$$\phi_n \longrightarrow 0 \quad \text{a.e. in} \quad L^1(\Omega),$$

what implies that

$$\lim_{n \to \infty} \int_\Omega \varphi(x)(g_n(x) - g(x)) dx = 0, \ \forall \varphi \in F,$$

and by density of F in $L^{r'(x)}(\Omega, \gamma^*)$, we conclude that:

$$\lim_{n \to \infty} \int_\Omega \varphi(x) g_n(x) dx = \int_\Omega \varphi(x) g(x) dx, \ \forall \varphi \in L^{r'(x)}(\Omega, \gamma^*),$$

that means

$$g_n \rightharpoonup g \text{ weakly in } L^{r(x)}(\Omega, \gamma).$$

■

Lemma 3 *Assume that (3.1), (3.3), (3.5) hold, let $(u_n)_n$ a sequence in $W_0^{1,p(x)}(\Omega, v)$ and $u \in W_0^{1,p(x)}(\Omega, v)$.*

$$If \begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega, v), \\ \text{and } \int_\Omega \Big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u) \Big)(\nabla u_n - \nabla u) dx \to 0 \end{cases}$$

then $u_n \longrightarrow u$ strongly in $W_0^{1,p(x)}(\Omega, v)$.

Proof:
Let us pose

$$D_n = \Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big)(\nabla u_n - \nabla u),$$

according the (3.5), we have D_n is a positive function. We have also $D_n \to 0$ in $L^1(\Omega)$.
Seeing that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega,\nu),$$

we have then

$$u_n \longrightarrow u \text{ strongly in } L^{q(x)}(\Omega,\sigma),$$

and consequently

$$u_n \longrightarrow u \text{ a.e. in } \Omega.$$

Thanks to $D_n \to 0$ a.e. in Ω,
there is then $B \subset \Omega$ such that $mes(B) = 0$ and for $x \in \Omega \setminus B$, we have

$$|u(x)| < \infty,\ |\nabla u(x)| < \infty,\ u_n(x) \longrightarrow u(x) \text{ and } D_n(x) \to 0.$$

Let us pose

$$\xi_n = \nabla u_n(x),\ \ \xi = \nabla u(x),$$

we have

$$\begin{aligned} D_n(x) \ &\geq \alpha \sum_{i=1}^{N} \omega_i |\xi_n^i|^{p(x)} + \alpha \sum_{i=1}^{N} \omega_i |\xi^i|^{p(x)} \\ &- \sum_{i=1}^{N} \omega_i^{\frac{1}{p(x)}} [k(x) + \sigma^{\frac{1}{p'(x)}} |u_n|^{\frac{q(x)}{p'(x)}} \\ &+ \sum_{j=1}^{N} \omega_j^{\frac{1}{p'(x)}} |\xi_n^j|^{p(x)-1}]|\xi^i| \\ &- \sum_{i=1}^{N} \omega_i^{\frac{1}{p(x)}} [k(x) + \sigma^{\frac{1}{p'(x)}} |u_n|^{\frac{q(x)}{p'(x)}} \\ &+ \sum_{j=1}^{N} \omega_j^{\frac{1}{p'(x)}} |\xi^j|^{p(x)-1}]|\xi_n^i|, \\ &\geq \alpha \sum_{i=1}^{N} \omega_i |\xi_n^i|^{p(x)} - C(x)[1 + \sum_{j=1}^{N} \omega_j^{\frac{1}{p'(x)}} |\xi_n^j|^{p(x)-1} \\ &+ \sum_{i=1}^{N} \omega_i^{\frac{1}{p'(x)}} |\xi_n^i|] \end{aligned} \tag{13}$$

where $C(x)$ is a function depending on x and not on n.
seeing that $u_n(x) \longrightarrow u(x)$, we have $|u_n(x)| \leq M_x$, where M_x is positive. Then by a standard argument, we will have $|\xi_n|$ is uniformly bounded compared to n; indeed: (4.4) becomes

$$D_n(x) \ \geq \ \sum_{i=1}^{N} \omega_i |\xi_n^i|^{p(x)} (\alpha \omega_i \ - \ \frac{C(x)}{N|\xi^i|^{p(x)}} \ - \ \frac{C(x)\omega_i^{\frac{1}{p'(x)}}}{|\xi_n^i|} \ - \ \frac{C(x)\omega_i^{\frac{1}{p(x)}}}{|\xi_n^i|^{p(x)-1}}).$$

If $|\xi_n| \to \infty$, there is at least i_0 such that$|\xi_n^{i_0}| \to \infty$, what will give us $D_n(x) \to \infty$; what is absurd.
Let ξ^* an adherent point of ξ_n, we have $|\xi^*| < \infty$ and by continuity of the operator a compared to two last variables, we will have:

$$\Big(a(x,u,\xi^*) - a(x,u,\xi)\Big)(\xi^* - \xi) = 0,$$

and according to (3.2), we obtain $\xi^* = \xi$.
the unicity of the adherent point implies $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω.
Seeing that the sequence $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{p(x)}(\Omega,\nu))^N$, and $a(x,u_n,\nabla u_n) \longrightarrow a(x,u,\nabla u)$ a.e. in Ω, then according the lemma 2 we obtain

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p(x)}(\Omega,\nu))^N.$$

let us pose $\widetilde{y_n} = a(x,u_n,\nabla u_n)\nabla u_n$ and $\widetilde{y} = a(x,u,\nabla u)\nabla u$, in the same way that in [4], we can write
$\widetilde{y_n} = a(x,u_n,\nabla u_n)\nabla u_n \longrightarrow \widetilde{y} = a(x,u,\nabla u)\nabla u$ strongly in $L^1(\Omega)$.
According to (3.3), we have

$$\alpha \sum_{i=1}^{N} \nu |\frac{\partial u_n}{\partial x_i}|^{p(x)} \leq \widetilde{y_n}.$$

let

$$z_n = \sum_{i=1}^{N} \nu |\frac{\partial u_n}{\partial x_i}|^{p(x)},\ \ z = \sum_{i=1}^{N} \nu |\frac{\partial u}{\partial x_i}|^{p(x)},\ \ y_n = \frac{\widetilde{y_n}}{\alpha},\ \ y = \frac{\widetilde{y}}{\alpha}.$$

Thanks to Fatou lemma, we obtain

$$\int_\Omega 2y\,dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|dx,$$

ie

$$0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|dx,$$

then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|dx \leq \limsup \int_\Omega |z_n - z|dx \leq 0,$$

what implies

$$\nabla u_n \longrightarrow \nabla u \text{ in } (L^{p(x)}(\Omega,\nu))^N,$$

consequently

$$u_n \longrightarrow u \text{ in } W_0^{1,p(x)}(\Omega,\nu).$$

■

Definition 3 *for any* $k > 0$ *and* $s \in I\!R$, *the truncature function* $T_k(.)$ *is defined as:*

$$T_k(s) = \begin{cases} s & if \quad |s| \leq k, \\ k\dfrac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Lemma 4 *Let* $(u_n)_n$ *a sequence from* $W_0^{1,p(x)}(\Omega,\nu)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p(x)}(\Omega,\nu)$. *Then* $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^{1,p(x)}(\Omega,\nu)$.

Proof
We have $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$, and $W_0^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we will have $u_n \to u$ strongly in $L^{p(x)}(\Omega)$ and a.e. in Ω, consequently $T_k(u_n) \to T_k(u_n)$ a.e. in Ω.
On another side

$$\begin{aligned} \|T_k(u_n)\|_{p(x),\nu}^{\theta_1} &\leq \int_\Omega |\nabla T_k(u_n)|^{p(x)}\nu(x)dx, \\ &\leq \int_\Omega |\nabla T_k'(u_n)||\nabla u_n|^{p(x)}\nu(x)dx, \\ &\leq \int_\Omega |\nabla u_n|^{p(x)}\nu(x)dx, \\ &\leq \|u_n\|_{p(x),\nu}^{\theta_2}, \end{aligned}$$

where

$$\theta_1 = \begin{cases} p^+ & \text{if } \|T_k(u_n)\|_{p(x),\nu} \leq 1, \\ p_- & \text{if } \|T_k(u_n)\|_{p(x),\nu} > 1, \end{cases}$$

and

$$\theta_1 = \begin{cases} p^+ & \text{si } \|u_n\|_{p(x),\nu} \geq 1, \\ p_- & \text{si } \|u_n\|_{p(x),\nu} < 1. \end{cases}$$

Thus $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, consequently $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$. ■

6 Proof of theorem 1

Step1: a priori estimate
The problem $(\mathcal{P}_n)$ admits at least a solution u_n belonging to $W_0^{1,p(x)}(\Omega,\nu)$. Choosing $T_k(u_n)$ as test function in $(\mathcal{P}_n)$, and seeing that $\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \geq 0$, we obtain:

$$\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx \leq \int_\Omega f_n T_k(u_n)dx.$$

Using (5), we deduce that

$$\alpha \int_\Omega \nu(x)|\nabla T_k(u_n)|^{p(x)}dx \leq kC_1,$$

that means

$$\int_\Omega \nu(x)|\nabla T_k(u_n)|^{p(x)}dx \leq \frac{k}{\alpha}C_1.$$

Consequently

$$\|\nabla T_k(u_n)\|_{p(x),\nu}^{\gamma} \leq C_2,$$

where

$$\gamma = \begin{cases} p^+ & if \quad \|\nabla T_k(u_n)\|_{p(x),\nu} \geq 1, \\ p_- & if \quad \|\nabla T_k(u_n)\|_{p(x),\nu} < 1. \end{cases}$$

Onthe other hand, we have

$$\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx.$$

What implies

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \leq \int_\Omega |f_n|kdx.$$

Thus

$$\int_{\{|u_n|>k\}} k\frac{u_n}{|u_n|}g_n(x,u_n,\nabla u_n)dx + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)u_n dx \leq \int_\Omega |f_n|kdx.$$

Consequently

$$k\int_{\{|u_n|>k\}} g_n(x,u_n,\nabla u_n)dx \leq kC_1$$

However

$$\int_\Omega \nu(x)|\nabla u_n|^{p(x)}dx = \int_{\{|u_n|>k\}} \nu(x)|\nabla u_n|^{p(x)}dx + \int_\Omega \nu(x)|\nabla u_n|^{p(x)}dx.$$

Then for $k > \rho_1$, we will have:

$$\int_\Omega \nu(x)|\nabla u_n|^{p(x)}dx \leq \frac{1}{\rho_2}\int_{\{|u_n|>k\}} g_n(x,u_n,\nabla u_n)dx + C_2 \leq C_4$$

What implies that a sequence $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$.
Step2: Strong convergence of truncations
Seeing that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, and $W_0^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we can extract a subsequence of $(u_n)_n$, still noted $(u_n)_n$, and there is $u \in W_0^{1,p(x)}(\Omega,\nu)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in} \quad W_0^{1,p(x)}(\Omega,\nu), \\ u_n \to u & \text{a.e. in } \Omega, \end{cases}$$

We want to show that $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p(x)}(\Omega,\nu)$.
Let $z_n = T_k(u_n) - T_k(u)$, let us pose $v_n = \varphi_\lambda(z_n)$, where $\varphi_\lambda(s) = se^{\lambda s^2}$.
Choosing v_n as test function in $(\mathcal{P}_n)$.
We are $(v_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, and $v_n \to 0$ a.e. in Ω, then according to lemma 4, we obtain $v_n \rightharpoonup 0$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$,.
thus $\langle f_n, v_n\rangle \to 0$, because $v_n \rightharpoonup 0$ weakly in $L^\infty(\Omega)$, and $f_n \to f$ strongly in $L^1(\Omega)$.
Consequently

$$\eta_{1n} = \langle Au_n, v_n\rangle + \langle G_n u_n, v_n\rangle \longrightarrow 0.$$

Seeing that $g_n(x,u_n,\nabla u_n) \geq 0$ on $\{x \in \Omega : |u_n| \geq k\}$, then we have:

$$\langle Au_n, v_n\rangle + \int_{\{x\in\Omega:|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)v_n dx \leq \eta_{1n}.$$

Thus

$$\langle Au_n, v_n\rangle - |\int_{\{x\in\Omega:|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)v_n dx| \leq \eta_{1n}. \quad (14)$$

however

$$\langle Au_n, v_n\rangle = \int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx$$
$$= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx$$
$$-\int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'_\lambda(z_n)dx \quad (15)$$

$= \int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx + \eta_{2n}$,
where
$\eta_{2n} = \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx - \int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'_\lambda(z_n)dx.$
We have

$$\begin{cases} \nabla T_k(u)\chi_{\{|u_n>k|\}} \to 0 \text{ strongly in } (L^{p(x)}(\Omega,\nu))^N, \\ (a(x,u_n,\nabla u_n))_n \text{ is bounded in } (L^{p'(x)}(\Omega,\nu^*))^N, \end{cases}$$

$\Rightarrow \eta_{2n} \longrightarrow 0$ when $n\to\infty$.
On another side
$|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)v_n dx|$
$\le \int_{\{|u_n|\le k\}} |g_n(x,u_n,\nabla u_n)||v_n|dx$

$$\le \int_{\{|u_n|\le k\}} d(k)(c(x)+\nu(x)|\nabla u_n|^{p(x)}|v_n|dx$$
$$\le d(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_\lambda(z_n)|dx \quad (16)$$
$$+\frac{d(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n|\varphi_\lambda(z_n)|dx$$

$\le \eta_{3n} + \frac{d(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_\lambda(z_n)|dx$
$\le \frac{d(k)}{\alpha}\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u_n))\Big)$
$\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]|\varphi_\lambda(z_n)|dx + \eta_{4n}$,
where

$$\eta_{3n} = d(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_\lambda(z_n)|dx \to 0 \text{ when } n\to\infty,$$

and $\eta_{4n} = \frac{d(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi_\lambda(z_n)|dx$
$+\frac{d(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))|\varphi_\lambda(z_n)|dx$
$+\eta_{3n} \to 0$ when $n\to\infty$,
If $\lambda \ge (\frac{d(k)}{\alpha})^2$ then, by a simple calculation, we have

$$\varphi'_\lambda(s) - \frac{d(k)}{\alpha}|\varphi_\lambda(s)| \ge \frac{1}{2}.$$

By combining this last inequality with (14), (15) and (16), we obtain
$\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u_n))\Big)$
$\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]$
$\le 2(\eta_{1n}-\eta_{2n}+\eta_{4n}), \longrightarrow 0$; when $n\to\infty$.
Thanks to lemma 3, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \text{ strongly in } W_0^{1,p(x)}(\Omega,\nu). \quad (17)$$

step3: Equi-integrability of the nonlinearitie $(g_n(x,u_n,\nabla u_n))_n$
We will show that $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$. Seeing (17), we deduce that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega, \text{ when } n\to\infty. \quad (18)$$

Let E a measurable part of Ω, and let $m>0$, let us pose

$$X_m^n = \{x\in\Omega : |u_n|\le m\} \text{ et } (X_m^n)^c = \{x\in\Omega : |u_n|>m\}.$$

We have

$$\begin{aligned}\int_E |g_n(x,u_n,\nabla u_n)|dx &= \int_{E\cap X_m^n} |g_n(x,u_n,\nabla u_n)|dx \\ &+ \int_{E\cap (X_m^n)^c} |g_n(x,u_n,\nabla u_n)|dx, \\ &\le d(m)\int_E (\nu(x)|\nabla T_m(u_n)|^{p(x)} \\ &+c(x))dx + \int_{(X_m^n)^c} |g_n(x,u_n,\nabla u_n)|dx.\end{aligned} \quad (19)$$

Thanks to (17) and seeing $c(x)\in L^1(\Omega)$, there is $\delta>0$ such that for all E: $|E|<\delta$, then

$$d(m)\int_E (\nu(x)|\nabla T_m(u_n)|^{p(x)} + c(x))dx \le \frac{\eta}{2}. \quad (20)$$

Let ψ_m a function defined by:

$$\psi_m(x) = \begin{cases} 0 & \text{if } |s|\le m-1, \\ 1 & \text{if } s\ge m, \\ 1 & \text{if } s\le -m, \\ \psi'_m(x)=1 & \text{if } m-1\le |s|\le m, \end{cases}$$

we have $\psi_m(u_n)\in W_0^{1,p(x)}(\Omega,\nu)$, choosing $\psi_m(u_n)$ as test function in $(\mathcal{P}_n)$, we obtain:
$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\psi'_m(u_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)\psi_m(u_n)dx$
$= \int_\Omega f_n\psi_m(u_n)dx.$
According (9) and (10), we deduce that

$$\int_{\{|u_n|>m-1\}} |g_n(x,u_n,\nabla u_n)|dx \le \int_{\{|u_n|>m-1\}} |f_n|dx.$$

what implies

$$\int_{\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx \le \int_{\{|u_n|>m-1\}} |f_n|dx.$$

however $f_n\to f$ in $L^1(\Omega)$ and $|\{|u_n|>m-1\}|\to 0$ when $m\to\infty$, uniformly in n, then for m big enough we have;

$$\int_{\{|u_n|>m-1\}} |f_n|dx \le \frac{\eta}{2}, \forall n\in I\!N.$$

Thus

$$\int_{\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx \leq \frac{\eta}{2}, \forall n \in I\!N. \quad (21)$$

Combining (19), (20) and (21), we deduce that

$$\int_E |g_n(x,u_n,\nabla u_n)|dx \leq \eta, \text{ for all } n \in I\!N.$$

That means that the sequence $(g_n(x,u_n,\nabla u_n))_n$ is equi-integrable, but $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ a.e. in Ω, then thanks to Vitali theorem, we deduce that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega).$$

step4: passing to the limit

Seeing that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, and thanks to (6), we conclude that $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{p'(x)}(\Omega,\nu^*))^N$, and seeing that $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, then according to the lemma 2, we obtain that

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p'(x)}(\Omega,\nu^*))^N,$$

while making $n \to \infty$, we obtain $\forall v \in W_0^{1,p(x)}(\Omega,\nu) \cap L^\infty(\Omega)$,

$$\langle Au, v\rangle + \int_\Omega g(x,u,\nabla u)v dx = \int_\Omega f v dx.$$

7 Example of application

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$, and let $p(x), q(x) \in C_+(\overline{\Omega})$.
Let us pose:

$$\begin{cases} a_i(x,r,\zeta) = \nu(x)|\zeta_i|^{p(x)-1}sgn(\zeta_i),\ i = 1,...,N, \\ \text{and} \\ g(x,r,\zeta) = \rho r|r|^{q(x)}\nu(x)|\zeta|^{p(x)}, \rho > 0, \end{cases}$$

were $\nu(x)$ a function weight in Ω. The function a_i, $i = 1,...,N$, which satisfies the assumptions of theorem (6), (7) and (8); and well as the function g satisfies (9), (10) and (11), with $|s| \geq \rho_1 = 1$ and $\rho_2 = \rho > 0$, consequently the assumptions of theorem 1 are satisfied; thus for $f \in L^1(\Omega)$, the following theorem, with $\nu(x) = d^\lambda(x)$:

$$(\mathcal{E})\begin{cases} \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p(x)-1}sgn(\frac{\partial u}{\partial x_i})\right) \\ +\rho u|u|^{q(x)}\sum_{i=1}^N \frac{\partial}{\partial x_i}d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p(x)} = f \text{ in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p(x)}(\Omega, d^\lambda(x)) \text{ and} \\ \rho u|u|^{q(x)}\sum_{i=1}^N \frac{\partial}{\partial x_i}d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p(x)} \in L^1(\Omega), \end{cases}$$

admits at least one solution $u \in W_0^{1,p(x)}(\Omega, d^\lambda(x))$.

References

1. Y. Akdim, E. Azroul and A. Benkirane, "Existence of solution for quasilinear degenerated elliptic equation.", Electronic J. Equ. **71**, 1-19, 2001.
2. E. Azroul, A. Barbara and H. Hjiej, "Strongly nonlinear p(x)-elliptic problems with second member L^1-dual.", African Diaspora Journal of Mathematics. **16**, Number 2, 11722, 2014.
3. E. Azroul, H. Hjiej and A. Touzani, "Existence and regularity of entropy solutions for strongly nonlinear p(x)-elliptic equations.", Electronic journal of differential equations. **68**, 1-27, 2013.
4. M. B. Benboubker, E. Azroul, A. Barbara, "Quasilinear Elliptic Problems with non standard growth.", EJDE, **62**, 1-16, 2011.
5. A. Benkirane and A. Elmahi, "Strongly nonlinear elliptic unilateral problems having natural growth terms and L^1 data.", Rendiconti di matematica, Serie VII, **18**, 28917303, 1998.
6. A. Bensoussan, L. Boccardo and F. Murat, "On a nonlinear partial differential equation having natural growth terms and unbounded solution.", Ann. Inst. Henri Poincaré, **4**, 347-364, 1988.
7. L. Boccardo and T. Gallouët, "Strongly nonlinear elliptic equations having natural growth terms and L^1.", Nonlinear Analysis Theory methods and applications, **19**(6), 573-579, 1992.
8. L. Boccardo and T. Gallouët, "Nonlinear elliptic equations with right hand side measures.", Comm. P.D.E., **17**, 641-655, 1992 .
9. H. Brezis and W. Strauss, "Semilinear second-order elliptic equations in L^1.", J. Math. Soc. Japan, **25**(4), 565-590, 1973.
10. T. Del Vecchio, "Nonlinear elliptic equations with measure data.", Potential Analysis, **4**, 185-203, 1995.
11. P. Drabek, A. Kufner and V. Mustonen, "Pseudo-monotonicity and degenerated or singular elliptic operators.", Bull. Austral. Math. Soc., **58**, 213-221, 1998.
12. X. L. Fan and Q. H. Zhang, "Existence for p(x)-Laplacien Direchlet problem.", Nonlinear Analysis, **52**, 1843-1852, 2003.
13. X. L. Fan and D. Zhao, "On the generalized Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$.", J. Gansu Educ. College, **12**(1), 1-6, 1998.
14. O. Kovácik and J. Rákosnik, "On Spaces $L^{p(x)}$ and $W^{k,p(x)}$.", Czechoslovak Math. J. , **41**, 592-618, 1991.
15. J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires.", Dunod, 1969.
16. V. M. Monetti and L. Randazzo, "Existence results for nonlinear elliptic equations with p-growth in the gradient.", Riceeche di Matimaica, **XLIX**(1), 163-181, 2000.
17. D. Zhao, X. J. Qiang and X. L. Fan, "On generalized Orlicz Spaces $L^{p(x)}(\Omega)$.", J. Gansu Sci., **9**(2), 1-7, 1997.