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Redlich-Kister Finite Difference Solution for Solving Two-Point Boundary Value Problems by using Ksor Iteration Family

Mohd Norfadli Suardi*, Jumat Sulaiman

Mathematics with Economics Programme, Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia

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ABSTRACT

In this paper, we are concerned to investigate the efficiency of the second-order Redlich-Kister Finite Difference (RKFD) discretization scheme together with the Four Point Explicit Group Kaudd Successive Over Relaxation (4EGKSOR) iterative method for solving two-point boundary value problems (TPBVPs). In order to apply this block iteration to solve any linear system, firstly we discretize all derivative terms via the second-order RKFD discretization scheme over the proposed problem in order to get the second-order RKFD approximation equation. Due to the main characteristics of the coefficient matrix for the generated linear system which are large-scale and sparse, the best choice for solving this linear system is using one of the iterative methods. Therefore, the formulation of the Kaudd Successive Over Relaxation method together with the Explicit Group iteration method mainly on the Four-Point Explicit Group Kaudd Successive Over Relaxation (4EGKSOR) iterative method has been presented to solve this linear system iteratively. In order to show the efficiency of the 4EGKSOR, another two iterative methods have also been considered which are the Gauss-Seidel (GS) and the Kaudd Successive Over Relaxation (KSOR) to solve three examples of the proposed problems in which all numerical results obtained were recorded based on the number of iterations, execution time and maximum norm. Based on the performance analysis, clearly, the 4EGKSOR iterative method shows substantiated improvement in terms of the number of iterations and execution time.

1. Introduction

The successful of the development of numerical techniques for boundary value problems has been growing rapidly in the past few decades. Many researchers give more attention to this problem numerically and show the capability of their numerical techniques in solving this problem especially TPBVPs; due to its application, this problem can be found in science, engineering and physics fields including optimal control, beam deflection and heat flow [1-4]. In initial works on obtaining the numerical solutions of TPBVPs, many authors attempted to achieve higher accuracy by using the various numerical methods. It was done either by suggesting the families of spline methods for solving second-order two-point boundary value problem, see in [5-9]. The basic approach of these methods is dividing the interval into subinterval and at the same time, the construction of spline, B-spline and extended B-spline in each subinterval considered. For achieving better accuracy, these methods are required to solve a system of equations. Another numerical method is to solve TPBVPs by developing the innovative method based on the Galerkin method see in [10-12]. Numerous numerical methods are used to solve the boundary value problems related TPBVPs for obtaining the approximation equation can be seen in [13-17].

Besides of using the above numerical methods to solve the proposed problem (1) as stated in the first paragraph, many studies were also introduced via the use of the concept of finite difference method (FDM). As a result, several numerical discretization schemes mainly in a family of the finite difference schemes have been proposed to form a new finite difference discretization scheme. For example, the standard finite difference [18], Chebyshev finite difference [19] and Rational Finite Difference [20] are imposed for solving TPBVPs. Clearly, the development of Chebyshev finite difference and Rational Finite Difference schemes has been encouraged by the combination of the standard finite difference concept together with the Chebyshev and rational approximation functions respectively. In conjunction with these combinations, the author in [20] also introduced a new finite difference scheme via the combination of exponential

*Corresponding Author: Mohd Norfadli Suardi, norfadli1412@gmail.com

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approximations and finite difference discretization schemes which is known as exponential finite difference discretization schemes to show its capability for solving TPBVPs [21].

Apart from the use of Chebyshev, rational and exponential approximation functions as mentioned in the previous second paragraph, this paper attempts to investigate the feasibility of the Redlich-Kister polynomial as a numerical method for solving TPBVPs. Based on the previous studies on the application of the Redlich-Kister polynomial, the findings have pointed out that various types of Redlich-Kister polynomial functions have successfully been used to develop the appropriate mathematical models in physics and chemistry fields [22-24]. In addition to these functions, only one study has been explored to investigate the application of the Redlich-Kister approximation function in the numerical analysis particularly on constructing the mathematical model. For instance, in [25], the authors investigated the construction of two mathematical models based on the piecewise third-order Redlich-Kister polynomial model and the piecewise first-order polynomial model respectively to show the relationship of the number of iterations for Gauss-Seidel towards its corresponding grid size. The findings of their work concluded that the results of the piecewise third-order Redlich-Kister polynomial model gave highly accurate solutions as compared with the first-order polynomial solution. Inspired by the high accuracy of the Redlich-Kister (RK) function based on high-order approximation function, we present a feasibility study of two newly established Redlich-Kister Finite Difference (RKFD) discretization schemes for solving TPBVPs.

In order to solve the proposed problem, firstly, the RKFD discretization scheme will be used to discretize TPBVPs to form the RKFD approximation equation. After that, the approximate equation was obtained will lead us to construct a linear system. Since the generated linear system has a large and sparse matrix, the use of iterative methods is the best linear solver [26-28]. For instance, the implementation of the point iteration family namely SOR [29], AOR [30] and KSOR [31] can be used to solve this linear system. In addition to this point iteration family, Evans [32] introduced the Explicit Group (EG) iterative method which is faster than the Gauss-Seidel (GS) iterative method to get the numerical solution of this linear system. Despite the speed up the convergence rate for Explicit Group (EG) iteration, many researchers also developed new variants of the EG iteration family such as 9-Point EG [33], EGSOR [34], EDGSOR [35] and MEGSOR [36] in which all these block iterations have significantly reduced their convergence rate. Therefore, further discussion of this paper focuses on investigating the efficiency of the 4EGKSOR iterative method which is inspired by the paper research [37] and apply together with the newly established RKFD discretization scheme for solving the system of Redlich-Kister approximation equations. The formulation of the 4EGKSOR iterative method can be established via the combination of the EG and KSOR iterative methods.

Before applying and investigating the performance of this 4EGKSOR method, we need to do the process of discretization and let us consider the general equation for TPBVPs, which are given as follows

$$\frac{d^2U}{dx^2}+Z(x)\frac{dU}{dx}+G(x)U(x)=r(x), \quad x\in[0,\phi] \tag{1}$$

with the Dirichlet conditions,

$$U(0)=\varphi_0, \qquad U(\phi)=\varphi_1.$$

2. Redlich Kister Finite Difference Approximation Equation

The previous section has mentioned that the use of the RK function to introduce two newly established RKFD discretization schemes for approximating the proposed problem (1). To start the discretization process, firstly let us consider the RK approximation function of order n as follows

$$U_n(x)=\sum_{k=0}^{n}a_k\cdot T_k(x) \tag{2}$$

where $a_k, k=0,1,2,...,n$ are the unknown parameters to be determined.

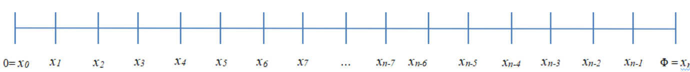

Figure 1: Distribution of grid network considered.

To facilitate us in discussing the use of this approximation function, let us construct the distribution of uniformly node points as indicated in Figure 1. Based on Figure 1, let us consider the first three RK functions that need to consider the number of node points as shown in Figure 2.


Figure 2: The path for T_1, T_2 and $_3$.

Then to derive the new second-order RKFD approximation equation, let us consider $n=2$ in equation (2) and applying the concept in Figure 2, the second-order RK approximation function can be written as

$$U(x)=a_0T_0(x)+a_1T_1(x)+a_2T_2(x) \tag{3}$$

where the first three RK functions are defined as

$$T_0(x)=1,$$
$$T_1(x)=x,$$
$$T_2(x)=x(1-x).$$

Referring to Figure 1, let us define the node points, $x_c=x_0+ch, c=0,1,2,...,n$ where $h=\frac{\phi-0}{n}, n=2^p, p\geq 1$ donates the uniform step size. Then $U(x_k)=U_k, k=c-1,c,c+1$ and $T(x_k)=T_k, k=c-1,c,c+1$ represent the approximation value of functions, $U(x)$ and $T(x)$. By considering any group of three node points, x_{c-1}, x_c and x_{c+1} for equation (3), we have the following equations

$$U_{c-1}=a_0T_{0,c-1}+a_1T_{1,c-1}+a_2T_{2,c-1}, \tag{4}$$

$$U_c = a_0T_{0,c} + a_1T_{1,c} + a_2T_{2,c}, \tag{5}$$

$$U_{c+1} = a_0T_{0,c+1} + a_1T_{1,c+1} + a_2T_{2,c+1}. \tag{6}$$

After that, the expression of three parameters $a_k, k = 0,1,2$ in equation (3) can be determined by solving the equations (4), (5) and (6) via a matrix approach. With these three parameters, we rewrite the second-order RK approximation function, $U(x)$ in equation (3) in which it can be shown as follows

$$U(x) = N_0(x)U_{c-1} + N_1(x)U_c + N_2(x)U_{c+1} \tag{7}$$

where the second-order RKFD shape functions, $N_k(x), k = 0,1,2$ are defined respectively as

$$\begin{cases} N_0(x) = \dfrac{1}{2h^2}\left(x^2\text{-}2xhc\text{-}xh + h^2c^2 + h^2c\right), \\ N_1(x) = \dfrac{1}{h^2}\left(2xhc\text{-}x^2\text{-}h^2c^2 + h^2\right), \\ N_2(x) = \dfrac{1}{2h^2}\left(x^2\text{-}2xhc + xh + h^2c^2\text{-}h^2c\right), \end{cases} \tag{8}$$

Then the first and second derivative of these RK shape functions can be shown as

$$\begin{cases} N_0'(x) = \dfrac{1}{2h^2}(2x\text{-}h\text{-}2hc), \\ N_1'(x) = \dfrac{1}{h^2}(2hc\text{-}2x), \\ N_2'(x) = \dfrac{1}{2h^2}(2x + h\text{-}2hc), \end{cases} \tag{9}$$

and

$$\begin{cases} N_0''(x) = \dfrac{1}{h^2}, \\ N_1''(x) = -\dfrac{2}{h^2}, \\ N_2''(x) = \dfrac{1}{h^2}. \end{cases} \tag{10}$$

Then, by applying the first derivative concept into the function (7) with respect to x_c, it can be shown that the second-order RKFD discretization scheme of the first derivative of the function, $U(x)$ is given as

$$\left.\frac{\partial U}{\partial x}\right|_c = N_0'(x_c)U_{c-1} + N_1'(x_c)U_c + N_2'(x_c)U_{c+1} \tag{11}$$

and for the second derivative of the function $U(x)$ with respect to x_c can be approximated by

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_c = N_0''(x_c)U_{c-1} + N_1''(x_c)U_c + N_2''(x_c)U_{c+1} \tag{12}$$

where $U(x_c) = U_{c,c=0,1,2,...,n}$ represent the approximation solution of function $U(x)$. Clearly, equations (11) and (12) are known as two newly established Redlich-Kister finite difference discretization schemes.

From the proposed problem (1), it needs to be rewritten in the discrete form at a node point, x_c and then we get

$$\left.\frac{d^2U}{dx^2}\right|_c + Z(x_c)\left.\frac{dU}{dx}\right|_c + G(x_c)U(x_c) = r(x_c), \tag{13}$$

Then by considering both equations (11) and (12) and substitute them into equation (13), it can be pointed out that the newly established RKFD approximation equation of TPBVPs can be formulated as follows

$$\alpha_cU_{c-1} + \beta_cU_c + \gamma_cU_{c+1} = r_c, \tag{14}$$

where

$$\alpha_c = N_0''(x_c) + Z_cN_0'(x_c),$$
$$\beta_c = N_1''(x_c) + Z_cN_1'(x_c) + G_c,$$
$$\gamma_c = N_2''(x_c) + Z_cN_2'(x_c).$$

and

$$Z_c = Z(x_c), G_c = G(x_c), r_c = r(x_c), \quad c = 1,2,3,...,n-1.$$

Referring to the RKFD approximation equation (14) and considering $c = 1,2,...,n-1$, it is obvious that we can construct a generated linear system with its large-scale and sparse coefficient matrix as follows

$$W \cdot U = r \tag{15}$$

where

$$W = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \\ 0 & 0 & 0 & 0 & \alpha_{n-1} & \beta_{n-1} \end{bmatrix},$$

$$U = \begin{bmatrix} U_1 & U_2 & U_3 & \cdots & U_{n-2} & U_{n-1} \end{bmatrix}^T,$$

$$r = \begin{bmatrix} r_1 - \alpha_1 \cdot \varphi_0 & r_2 & r_3 & \cdots & r_{n-2} & r_{n-1} - \gamma_{n-1} \cdot \varphi_1 \end{bmatrix}^T.$$

3. Derivation of 4EGKSOR Iterative Method

Since the characteristics of the coefficient matrix of the linear system (15) are large-scale and sparse, the family of iterative methods can be chosen to be the best linear solver as stated in the first section. To solve this linear system, the Kaudd Successive Over Relaxation (KSOR) iterative method was developed by [38] as one of more efficient point iterative methods by using one weighted parameter which is used to speed up its convergence rate and show to be more economical computationally than the Gauss-Seidel (GS) iterative method. Due to the advantage of lower computational complexity, we establish the formulation of the Four-Point Explicit Group Kaudd Successive Over Relaxation (4EGKSOR) iterative method, which is a combination between a standard Kaudd Successive Over Relaxation (KSOR) iterative method and Explicit Group approach by using the newly established RKFD approximation equation (14). To derive the formulation of 4EGKSOR, let us consider the grid network in Figure 1 and a group of block node points concept in Figure 3. Figure 3 illustrates the finite grid network of the RKFD approximation equation where block approach has been done until iteration convergence is achieved.

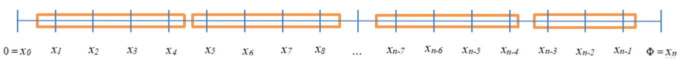

Figure 3: Distribution of grid network for 4EGKSOR method.

Before discussing more details on the formulation of the 4EGKSOR method, let us consider the coefficient matrix W (15) being defined as

$$W = F + J + L \tag{16}$$

where J is diagonal matrix, F and L are strictly lower and upper matrices of the generated linear system (15). Then, the large-scale and sparse linear system (15) becomes

$$(F + J + L)\cdot U = r \tag{17}$$

In drive to achieve the linear system (17) based on the point iteration approach, the implementation of the KSOR method over the linear system can be stated in matrix form as follows [31,39]

$$U^{(q+1)} = [(1-\omega)J - \omega F]^{-1}(J+L)U^{(q)} + [(1-\omega)J - \omega F]^{-1} r \tag{18}$$

where $U^{(q+1)}$ indicates the current value of U at the $(q+1)^{th}$ iteration.

Again, imposing the KSOR method (18) can also be rewritten in the point iteration approach as follows

$$U_c^{(q+1)} = \frac{1}{(1+\omega)} U_c^{(q)} + \frac{\omega}{(1+\omega)}\left(r_c - \alpha_c U_{c-1}^{(q+1)} - \gamma_c U_{c+1}^{(q)}\right) \tag{19}$$

for $c = 1,2,3,\ldots,n-1$, whereas the optimum value of ω the different value subject to the size value of n. The range value of ω is given [31] by $\omega \in R - [-2,0]$

By using the same steps to obtain equation (19), let us begin to introduce the KSOR block iteration approach. To start this discussion, let us consider again a sequence of a group of four and three node points in Figure 3. As we can see that two types of blocks were used to form during the implementation of the 4EGKSOR iteration. Firstly, the four-point block iteration is applied for the completed group of four and three point block is imposed into the ungrouped case. Referring to equation (15), the 4EGKSOR iterative method is defined as [37]

$$\begin{bmatrix} U_c \\ U_{c+1} \\ U_{c+2} \\ U_{c+3} \end{bmatrix}^{(q+1)} = \frac{1}{(1-\omega)} \begin{bmatrix} U_c \\ U_{c+1} \\ U_{c+2} \\ U_{c+3} \end{bmatrix}^{(q)} + \frac{\omega}{(1-\omega)} \begin{bmatrix} \beta_c & \gamma_c & 0 & 0 \\ \alpha_{c+1} & \beta_{c+1} & \gamma_{c+1} & 0 \\ 0 & \alpha_{c+2} & \beta_{c+2} & \gamma_{c+2} \\ 0 & 0 & \alpha_{c+3} & \beta_{c+3} \end{bmatrix}^{-1} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} \tag{20}$$

where

$$\begin{cases} S_1 = r_c - \alpha_c U_{c-1}, \\ S_2 = r_{c+1}, \\ S_3 = r_{c+2}, \\ S_4 = r_{c+3} - \gamma_c U_{c+1}, \end{cases}$$

Meanwhile, for the ungrouped case has been applied for only one group of three points block is shown as follows

$$\begin{bmatrix} U_{n-3} \\ U_{n-2} \\ U_{n-1} \end{bmatrix}^{(q+1)} = \frac{1}{(1-\omega)} \begin{bmatrix} U_{n-3} \\ U_{n-2} \\ U_{n-1} \end{bmatrix}^{(q)} + \frac{\omega}{(1-\omega)} \begin{bmatrix} \beta_{n-3} & \gamma_{n-3} & 0 \\ \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \\ 0 & \alpha_{n-1} & \beta_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} S_{n-3} \\ S_{n-2} \\ S_{n-1} \end{bmatrix} \tag{21}$$

where

$$\begin{cases} S_{n-3} = r_{n-3} - \alpha_i U_{n-4}, \\ S_{n-2} = r_{n-2}, \\ S_{n-1} = r_{n-1} - \gamma_n U_n, \end{cases}$$

Thus, Algorithm 1 describes a summary of the 4EGKSOR iterative method, which has been implemented for solving the proposed problem (1).

Algorithm 1: 4EGKSOR iteration

i. Set the initial value $U = 0$.
ii. Calculate the coefficient matrix, W.
iii. Calculate the vector, r.
iv. For $c = 1,5,9,\ldots,n-7$, calculate the equation (20).
v. For $c = n-3$, calculate the equation (21).
vi. Check the convergence test, $\left|U_c^{(q+1)} - U_c^{(q)}\right| \langle \varepsilon = 10^{-10}$.
   If yes, go to step (vii). Otherwise, go back to step (iv).
vii. Display approximate solution.

4. Numerical Problem and Discussion

In this section, we investigate the feasibility of the 4EGKSOR iterative method for solving three examples of one-dimensional TPBVPs and then compared with GS and KSOR methods which are set up as a benchmarking for this study. All the results that have been obtained by imposing these three methods considered into these three examples are analyzed based on three comparison parameters such as the number of iterations (Iter), execution time (Time) in seconds and maximum norm (MaxNorm) at five different sizes, $n = 256,512,1024,2048,4096$. In this study, we also set up the value of tolerance error, $\varepsilon = 10^{-10}$ for all grid sizes are considered.

Example 1 [40]

Consider one-dimensional TPBVPs as

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x} = -e^{(x-1)^{-1}}, \tag{22}$$

The analytical solution of problem (22) is $U(x) = x\left(1 - e^{(x-1)}\right)$.

Example 2 [41]

Consider one-dimensional TPBVPs

$$\frac{\partial^2 U}{\partial x^2}+xU(x)=\left(3-x-x^2+x^3\right)\sin(x)+4x\cos(x), \qquad (23)$$

The analytical solution of problem (23) is $U(x)=\left(x^2-1\right)\sin(x)$.

Example 3 [42]

Consider one-dimensional TPBVPs

$$\frac{\partial^2 U}{\partial x^2}+U(x)=-1, \qquad (24)$$

The analytical solution of problem (24) is

$$U(x)=\cos(x)+\frac{1-\cos(1)}{\sin(1)}\sin(x)-1.$$

All results of GS, KSOR and 4EGKSOR iterative methods in solving these three examples are stated in Tables 1, 2 and 3 and illustrated in Figure 4, 5, and 6 respectively. Table 4 shows that the reduction percentage of KSOR and 4EGKSOR iterative methods which compared with GS method for three examples considered.

Table 1: Numerical results based on comparison criteria considered for Problem 1.

n	Method	Iter	Time	MaxNorm
256	GS	82043.0	7.92	4.0343e-07
	KSOR	769.0	0.75	2.4866e-07
	4EGKSOR	364.0	0.17	2.3889e-07
512	GS	292276.0	16.23	2.5291e-06
	KSOR	1526.0	1.67	6.7370e-08
	4EGKSOR	759.0	0.44	8.0806e-08
1024	GS	1025489.0	76.67	1.0346e-05
	KSOR	2853.0	3.19	2.5732e-08
	4EGKSOR	1295.0	0.78	4.7925e-08
2048	GS	3527433.0	409.03	4.1443e-05
	KSOR	5792.0	6.63	1.7614e-08
	4EGKSOR	2545.0	1.50	5.9857e-08
4096	GS	11811519.0	2359.09	1.6579e-04
	KSOR	10221.0	10.41	9.8302e-08
	4EGKSOR	5369.0	3.13	7.2651e-08

Table 2: Numerical results based on comparison criteria considered for Problem 2.

n	Method	Iter	Time	MaxNorm
256	GS	92156.0	19.79	9.0029e-07
	KSOR	796.0	0.81	1.5811e-06
	4EGKSOR	414.0	0.30	1.5584e-06
512	GS	329819.0	58.89	2.4116e-06
	KSOR	1559.0	1.69	3.7928e-07
	4EGKSOR	756.0	0.39	4.0320e-07
1024	GS	1164082.0	257.58	1.1096e-05
	KSOR	3073.0	3.56	1.0466e-07
	4EGKSOR	1450.0	0.82	6.2177e-08
2048	GS	4035615.0	1345.67	4.4746e-05
	KSOR	6145.0	7.18	3.8252e-08
	4EGKSOR	2826.0	1.71	4.3353e-08
4096	GS	13659733.0	2913.87	1.7907e-04
	KSOR	11571.0	11.03	5.2215e-08
	4EGKSOR	5181.0	3.06	1.0216e-07

Table 3: Numerical results based on comparison criteria considered for Problem 3.

n	Method	Iter	Time	MaxNorm
256	GS	89973.0	19.88	5.4091e-07
	KSOR	782.0	0.37	1.9062e-07
	4EGKSOR	381.0	0.17	2.0338e-07
512	GS	318924.0	60.80	2.9059e-06
	KSOR	1537.0	0.89	5.2948e-08
	4EGKSOR	724.0	0.40	2.5886e-08
1024	GS	1111808.0	256.86	1.1810e-05
	KSOR	3057.0	1.82	1.5546e-08
	4EGKSOR	1387.0	0.79	6.1722e-08
2048	GS	3791677.0	1260.25	4.7285e-05
	KSOR	5734.0	3.43	2.5772e-08
	4EGKSOR	2753.0	1.75	8.8766e-08
4096	GS	12544476.0	2681.69	1.8915e-04
	KSOR	10655.0	5.78	1.0642e-07
	4EGKSOR	5463.0	3.21	1.1934e-07

Table 4: Reduction percentage for the KSOR and 4EGKSOR in term of the iteration and time.

n	Method	Iter	MaxNorm
Problem 1	Iter	99.06-99.84	99.56-99.95
	Time	89.71-99.56	97.29-99.87
Problem 2	Iter	99.14-99.92	99.55-99.96
	Time	95.91-99.62	98.48-99.89
Problem 3	Iter	99.13-99.92	99.57-99.96
	Time	98.13-99.92	99.14-99.96

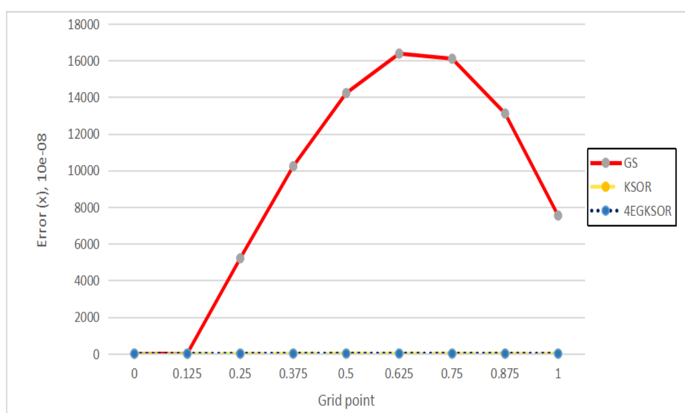

Figure 4: Comparison of three iterative methods based on error over the solution domain of Problem 1 at n = 4096.

All results are presented in Tables 1 to 4, the KSOR method gives reduced iteration and speeds up its execution time as compared with GS iterative method. Then, in terms of a reduction percentage, the KSOR iteration in Example 1 has significantly reduced number of iterations approximately by 99.06-99.84% and speed up by 89.71-99.56%. For Examples 2 and 3 also give the pattern as same as Example 1 in which KSOR iteration is better than GS iteration. However, the 4EGKSOR iteration gives tremendously improved either in the number of iterations or execution time which are 99.56-99.95% and 97.29-99.87% for Example 1, 99.55-99.96% and 98.48-99.89% for Example 2 and 99.57-99.96% and 99.14-99.96% for Example 3 respectively. In conclusion, it shows that the KSOR iteration has greatly reduced its number of iterations and execution time as compared to the GS iteration. It means that the 4EGKSOR iteration has the least amount compared to GS and KSOR iterations in terms of the number of iterations and execution time. In addition to these findings, for the maximum norm, KSOR and 4EGKSOR iterative methods show a good agreement and become close to their analytical solution compared to GS iterative method, see in Figures 4, 5 and 6.

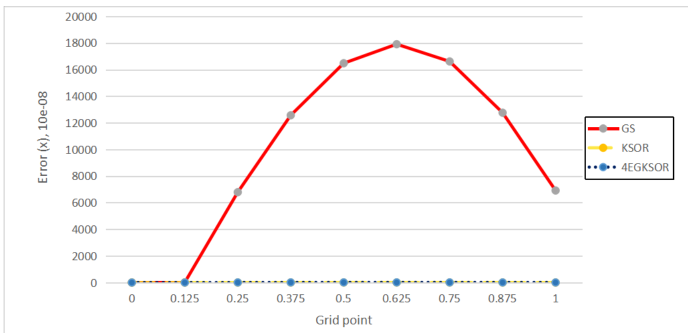

Figure 5: Comparison of three iterative methods based on error over the solution domain of Problem 2 at n = 4096.

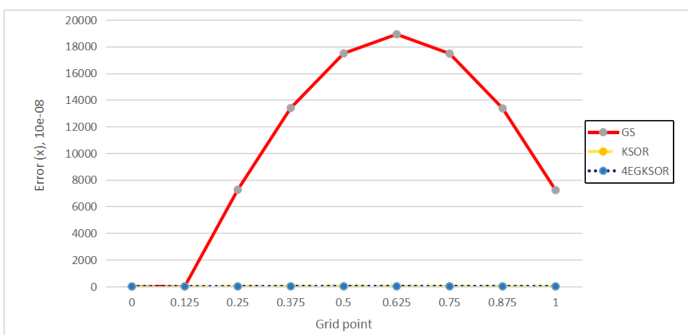

Figure 6: Comparison of three iterative methods based on error over the solution domain of Problem 3 at n = 4096.

5. Conclusions

The formulation of GS, KSOR and 4EGKSOR iterative methods have been successfully derived by using two newly established RKFD discretization schemes for solving TPBVPs. Then, the generated large-scale and sparse linear system based on two newly established RKFD approximation equations have been solved by using three iterative methods in which all results were recorded. Based on the implementation of these three iterative methods, the 4EGKSOR iterative method has tremendously reduced the iteration and Time as compared with GS and KSOR methods. Therefore, the combination of the KSOR block technique with the standard EG iterative method can reduce iteration and Time compared to the KSOR point approach. For further study, this paper can be extended to perform the use of the newly established RKFD discretization scheme for solving the multi-dimensional boundary value problems by using the two-step iteration family [43,44], and the half-sweep [45,46] and quarter-sweep [47,48] iteration families.

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